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Mathematics. — *"The Congruence of the Twisted Cubics which Pass through Four Given Points and which Cut Two Given Straight Lines."* By Dr. G. SCHAAKE. (Communicated by Prof. JAN DE VRIES).

(Communicated at the meeting of March 27, 1926).

§ 1. The twisted cubics k^3 which pass through four given points $H_1, \ldots, H_4$ and which cut two given lines l and m, may be represented on the points of a plane α in the following way. We choose two points L and M in α. We consider a projective correspondence between the points P of l and the lines p of the plane pencil (L, α) and a similar correspondence between the points Q of m and the lines q of the plane pencil (M, α). To a curve k^3 which cuts l in P and m in Q, we associate the point of intersection K of the lines p and q corresponding resp. to P and Q.

§ 2. To the line LM, which belongs to the plane pencil (L, α) as well as to the plane pencil (M, α), there correspond on l and m resp. the points A and B. The curve k^3 through A and B has all the points of LM as image points.

Accordingly the curve k^3 through A and B is singular for our representation. To this curve all the points of LM correspond as image points K.

If we choose K in L, the line p is indefinite and q coincides with LM. The point L is, therefore, the image point of the ∞^1 curves k^3 through B. These curves form a surface which cuts the plane $H_2 H_3 H_4$ along the lines $H_2 H_3$, $H_3 H_4$, $H_2 H_4$, and the conic through H_2, H_3, H_4, and the points of intersection of BH_1 and l with the said plane. For this conic, together with BH_1, forms a degenerate k^3 through B and $H_3 H_4$ forms a similar k^3 with the conic in the plane BH_1H_2 which cuts H_3H_4 and which passes through B, H_1, H_2 and the point of intersection of l with this plane. The surface has such an intersection with any plane through three of the points B and $H_1, \ldots, H_4$. The surface in question, i. e. the locus of the twisted cubics which pass through five given points and which cut a given line, is, accordingly, a surface of the fifth degree β^5 which has triple points in the points B and $H_1, \ldots, H_4$.

In the same way it appears that to the point M ∞^1 curves k^3 are associated, the curves k^3 through A, which also form a surface of the fifth degree, α^5.

Consequently the points L and M are singular for our representation. To L and M there correspond resp. the surfaces of the fifth degree β^5 and α^5 of the curves k^3 through B and A.

The conic k^2 passing through H_2, H_3 and H_4 and cutting l and m, which, therefore, contains the points of intersection P_1 and Q_1 of l and m with the plane $H_2 H_3 H_4$, together with the generatrices of the cone $\varkappa_1$ which projects $k_1{}^2$ out of H_1, forms ∞^1 curves k^3 of our system. All these curves are represented in the point of intersection S_1 of the lines p_1 and q_1 corresponding resp. to P_1 and Q_1.

Accordingly we find four singular image points S_1, S_2, S_3, and S_4. To S_i there correspond the ∞^1 curves k^3 that are formed by the conic k_i^2 cutting l and m resp. in P_i and Q_i, which passes through H_k, H_l, and H_m, and the generatrices of the cone $\varkappa_i$ which projects k_i^2 out of H_i.

The transversal t_1 of l and m through H_1 which cuts l and m resp. in P_1' and Q_1', forms ∞^1 curves k^3 with the conics of the pencil β_i of conics which cut t_1 and which pass through H_2, H_3, and H_4. All these curves k^3 are represented on the point of intersection S_1' of the lines p_1' and q_1' corresponding resp. to P_1' and Q_1'.

Consequently there are four more singular image points S_1', S_2', S_3', and S_4'. To S_i' there correspond the ∞^1 curves k^3 that are formed by the transversal t_i through H_i' of l and m, which cuts l and m resp. in P_i' and Q_i', with the conics of the pencil β_i of the conics that pass through H_k, H_l, and H_m and cut t_i.

The lines $H_1 Q_2$ and $H_2 P_1$ both cut the line $H_3 H_4$ and form a k^3 of our congruence together with this line. It is represented in the point of intersection K_{12} of p_1 and q_2. The points $K_{ik} \equiv p_i q_k$ $(i \neq k)$ are, therefore, the image points of twelve curves k^3 which are degenerate in a join of two of the points H_i and two lines which cut this line.

The degenerate curves k^3 consisting of a conic through H_2, H_3, H_4, P_1 or Q_1, and a transversal through H_1 of such a conic and m or l, are represented in the points of p_1 or of q_1. Besides the curves that are represented in the points $p_1 q_2, p_1 q_3$, and $p_1 q_4$, the system corresponding to p_1 contains three more curves k^3 degenerate in three lines e.g. the curve which consists of $H_3 H_4, H_2 P_1$, and the transversal through H_1 of $H_2 P_1$ and m. In this way we find $8 \times 3 = 24$ more curves k^3 degenerate in three lines.

There are twelve more of such curves, those which consist of the transversal t_i and the two lines of a degeneration in β_i, of which the image points, therefore, coincide in the points S_i' in groups of three.

Finally we find six curves k^3 degenerate in three lines, in the following way. Let us consider e.g. the curves k^3 consisting of $h_{12} \equiv H_1 H_2$ and a conic k^2 through H_3 and H_4 which rests on h_{12}, l and m. These conics k^2 form a surface of the fourth degree, a dimonoid with H_3 and H_4 as triple points and $h_{34} \equiv H_3 H_4$ as double torsal line. This arises from the pair of lines formed by h_{34} with one of the two transversals

of h_{34}, h_{12}, l and m. Each of these two transversals forms, therefore, with h_{12} and h_{34} a curve k^3 degenerate in three lines, so that in all our congruence contains six more figures k^3, each consisting of three lines.

The system of the curves k^3 which form the above mentioned monoid, is represented on a conic through L and M, for the point ranges (P) and (Q) of these curves are projective. This conic contains the points S_1 and S_2. Analogously to the dimonoid which is formed by the conics passing through H_1 and H_2 and cutting l, m and h_{34}, there corresponds a conic through L, M, S_3 and S_4.

Outside l the two conics have two points in common, the image points of the curves k^3 which consist of a transversal of l, m, h_{12} and h_{34} and the latter two lines.

The contents of the preceding three paragraphs have been communicated to me by Prof. JAN DE VRIES.

Consequently our congruence contains in all 54 curves degenerate in three lines.

§ 3. A curve k^3 which cuts l twice, has two image points, which lie on the same line q through M. The locus of the image points of these curves cuts any line q twice outside M in the image points of the curve k^3 that cuts l twice and that passes through the corresponding point Q. As l cuts the surface a^5 twice outside the triple point A, two of the curves k^3 through A cut l once more. Hence the locus of the image points of the curves k^3 which cut l twice, is a biquadratic curve λ^4 that has a double point in M.

λ^4 has also a double point in L. The tangents to λ^4 at this point are the two lines p corresponding to the points of intersection P with l of the curve k^3 through B which cuts l twice. Accordingly a line p cuts λ^4 twice outside L. Through a point of l there pass, therefore, just as through A, two curves k^3 which cut l once more.

The curve k^3 consisting of the conic k_i^2 and the line through H_i which cuts k_i^2 and l in different points and which is, accordingly, represented in S_i, cuts l twice, the same as the curve k^3 formed by t_i and the conic cutting l of the pencil β_i, of which the image point lies in S_i'. Consequently λ^4 passes through the points S_i and S_i'.

The points of contact of the four tangents to λ^4 through M are the image points of the four twisted cubics which pass through the four points H_i, touch l, and cut m.

The locus of the image points of the curves k^3 which cut m twice, is a biquadratic curve μ^4 which has double points in L and M and passes through the points S_i and S_i'. Each of the said curves k^3 has two image points, the two points of intersection outside L of μ^4 and the line p associated to its point of intersection with l.

Four of the sixteen points of intersection of λ^4 and μ^4 lie in each of the points L and M; the remaining eight are the points S_i and S_i'.

Hence no twisted cubic which cuts the lines l and m twice, passes through the points H_i.

§ 4. We shall now examine the representation of the system Σ_1 of the curves k^3 which cut an arbitrary line n. The curves k^3 passing through a point P of l or a point Q of m, form a surface of the fifth degree, which has been proved for the point B in § 2. Hence through any point of l or m there pass five curves of Σ_1. Consequently the image curve of Σ_1 has five points, resp. outside L and M, in common with each line p and q. As each of the systems associated to L and M contains five curves which cut n, the said image curve has fivefold points in L and M. Accordingly Σ_1 is represented in a curve of the tenth order, k^{10}, which has fivefold points in L and M.

Of the curves k^3 associated to S_i, two belong to Σ_1, those which are formed by k_i^2 and the two transversals of k_i^2 and n through H_i. Among the curves k^3 corresponding to S_i there is only one which belongs to Σ_1, the one which consists of t_i and the conic of the pencil β_i which cuts n. Hence the curve k^{10} has a double point in each of the points S_i and passes through all the points S_i'.

Ten of the fourty points of intersection of the curves k^{10} and λ^4 coincide in each of the points L and M and two in each of the points S_i, and also the points S_i' belong to them. The remaining eight points of intersection are yielded by the four pairs of image points of the curves k^3 which cut l twice and n once.

As, accordingly, through four points there pass four twisted cubics which cut a given line twice and two other given lines once, Σ_1 contains four curves which cut n twice. The image points of these curves are double points of k^{10}.

Consequently the image curve of the system Σ_1 of the curves k^3 which cut a given line n, is a curve k^{10} of the order ten and the genus eight which has the points L and M as fivefold points, has double points in the points S_i, passes through the points S_i', and has four more free double points.

25 of the hundred points of intersection of two curves k^{10} lie in each of the points L and M, four in each of the points S_i, and also the points S_i' belong to the points of intersection. There remain, therefore, thirty points of intersection that are not singular for the representation.

Accordingly there are thirty twisted cubics which pass through four given points and cut four given lines. [1])

If n cuts the line l, the line p corresponding to the point of intersection of l and n splits off from k^{10} and there remains a curve of the ninth order which has a quadruple point in L and a fivefold point in M. If n is a transversal of l and m, the image curve is of the eighth order and has quadruple points in L and M.

[1]) Cf. SCHUBERT, *Kalkül der abzählenden Geometrie*, p. 172.

If n passes through H_1, the image points of the curves k^3 which cut n outside H_1, form a curve which cuts any line p or q resp. outside L or M in two points. For n cuts two of the curves k^3 through a point of l or m, which form a surface of the fifth degree with triple point in H_1, outside H_1.

If we choose the point of l or m in A or B, it appears that the image curve has double points in L and M and is, therefore, of the fourth order. Among the curves k^3 corresponding to S_1 there is none which cuts n outside H_1. Of the curves k^3 associated to S_2 one cuts n outside H_1, to wit the curve which consists of k_2^2 and the line through H_2 which cuts n and k_2^2 in different points. As an analogous remark may be made for S_3 and S_4, the image curve k_1^4 does not pass through S_1, but it passes through S_2, S_3 and S_4. The curve k^3 consisting of t_1 and the conic of the pencil β_1 cutting n, which, accordingly, cuts n outside H_1, corresponds to S_1', and the curves k^3 associated to S_2', S_3', and S_4' do not contain any curve which cuts n outside H_1. Consequently the image curve k_1^4 passes through S_1', but not through S_2', S_3', and S_4'.

The curves k^3 which cut a line through H_i outside H_i, are represented on the points of a biquadratic curve k_i^4 with double points in L and M which passes through the points S different from S_i and through S_i'.

If n approaches to a line through H_1, the system of the curves k^3 which cut n, ends by degenerating into the system of the curves k^3 which cut the limitative position of n outside H_1 and the system of the curves k^3 which touch the limitative position of the plane $H_1 n$ at H_1.

Hence the curves k^3 which touch a plane through H_i at H_i, are represented on the points of a curve of the sixth order k_i^6 with triple points in L and M, which has a double point in S_i and passes through the points S and S' differing resp. from S_i and S_i'.

This may also be derived directly.

The points of intersection of a curve k_i^4 and a curve k^{10} are the points L and M, each counted ten times, the three points S different from S_i, each counted twice, the point S_i', and thirteen points that are not singular for the representation. Hence a straight line through H_i cuts thirteen of the curves k^3 which intersect an arbitrary line outside H_i.

Consequently the surface of the curves k^3 which cut an arbitrary straight line n, has a seventeenfold point in each point H_i. This surface, which is of the degree thirty, has l, m, and n as fivefold straight lines.

The sixteen points of intersection of two curves k_i^4 contain the points L and M, each counted four times, the three points S different from S_i, and the point S_i'. The other four points of intersection are not singular for the representation. A curve k_i^4 has six non-singular points in common with a curve k_k^4; for the singular points of intersection are the points L and M, each counted four times, and the two points S different from S_i and S_k.

Hence four of the curves k^3 cutting a line n through H_i outside H_i also cut another line through H_i outside H_i, and six cut a line through H_k outside H_k.

The surface of the thirteenth degree which is formed by the curves k^3 that cut a line through H_i outside H_i, has, therefore, a ninefold point in H_i and sevenfold points in the other points H_i. This surface has the lines l and m as double lines and n as a fivefold line.

By considering the representation of the system of the curves k^3 which touch a plane through H_i at H_i, or by observing that the surface formed by these curves completes the surface of the curves k^3 that cut a line through H_i in the said plane outside H_i, to an individual of the system of surfaces, each consisting of the curves k^3 which cut an arbitrary line, we find:

The curves k^3 which touch a plane through H_i at H_i, form a surface of the degree seventeen which has an eightfold point in H_i and tenfold points in the other points H. l and m are triple lines of this surface. For the tangents at H_i to the curves k^3 which pass through a point of l or m, form a cubic cone.

Four of the sixteen points of intersection of a curve λ^4 or μ^4 and a curve k_i^4 lie in each of the points L and M and also the points S different from S_i and the point S_i' belong to them. The other four non-singular points of intersection form the pairs of image points of two curves k^3 that cut l or m twice and that cut the line through H_i corresponding to k_i^4 outside H_i. Accordingly the biquadratic surface of the curves k^3 which cut l or m twice, have double points in the points H_i.

§ 5. A straight line a of α is the image of a system of curves k^3 which form a surface of the tenth degree Ω^{10}, for to the ten points of intersection of a and a curve k^{10} there correspond as many curves of the said system, which cut the line n associated to k^{10}. To the point of intersection of a and LM there corresponds the curve k^3 through A and B, which, therefore, lies on Ω^{10}. As the pairs of rays (p, q) with points of intersection on a define a projective correspondence between the plane pencils L and M, Ω^{10} is the locus of the curves k^3 which contain the pairs of points of a projective correspondence between the lines l and m.

As a has four points in common with each of the curves λ^4 and μ^4, Ω^{10} contains four curves k^3 which cut l twice and as many curves k^3 which cut m twice. To the six points of intersection of a with a curve k_i^6 there correspond as many curves k^3 of Ω^{10}, which touch the plane through H_i corresponding to k_i^6 at H_i. Accordingly Ω^{10} has a sextuple point in each point H_i. Besides l and m the surface contains eight more lines, the four lines which each join a point H_i to the point on l associated to the point of intersection of m with the plane through the other three points H, and the four lines which each pass through a

point H_i and through the point on m corresponding to the point of intersection of l with the plane through the other three points H. For each of these lines, together with a conic, forms a curve k^3 of Ω^{10} (cf. § 2).

A surface which is formed by the curves k^3 containing the pairs of a projective correspondence between l and m to which the pair (A, B) does not belong, is represented on a conic through L and M.

§ 6. The image curve of the system Σ_2 of the curves k^3 which touch a given plane φ, has as many points outside L in common with a straight line through L as curves k^3 touching φ pass through a point P of l. Through a cubic involutory transformation [1]) with cardinal points in $H_1,\ldots,H_4$ P is transformed into the point P', m into a cubic m'^3 through $H_1,\ldots,H_4$ and φ into a cubic surface φ'^3 which has conical points in $H_1,\ldots,H_4$. To the curves k^3 through P which touch φ, there correspond the tangents of φ'^3 through P which cut m'^3 outside $H_1,\ldots,H_4$. The tangents of φ'^3 through P' form a cone of the sixth degree which has the lines $P'H_1,\ldots,P'H_4$ as double generatrices. There are, accordingly, $6\times 3-4\times 2=10$ lines of the cone which cut m'^3 outside the points $H_1,\ldots,H_4$.

Consequently the image curve of Σ_2 has ten points outside L in common with any line through L. Through each of the points A and B there pass ten curves k^3 which touch φ. Hence the image curve of Σ_2 is a curve of the order twenty, k^{20}, with tenfold points in L and M.

As a pencil β_i contains two conics which touch φ and which with t_i form two curves k^3 associated to S_i', k^{20} has a double point in each of the points S_i'.

Among the curves k^3 through P_1, the point of intersection of l with the plane $H_2H_3H_4$, which touch φ, there are in the first place four curves which are formed by the two conics touching φ of the pencil with base points H_2, H_3, H_4, and P_1, each combined with the two transversals through H_1 of such a conic and the line m. Further we have the curve k^3 consisting of the conic through H_2, H_3, H_4, P_1 and the point of intersection of the planes $H_2H_3H_4$, φ and H_1m, and the join of the latter point of intersection with H_1. Finally we find the two curves k^3 formed by k_1^2 and the lines which project the points of intersection of k_1^2 and φ out of H_1. The latter two curves are represented on the point S_1. Now we have ten curves k^3 through P_1 which touch φ, if we count the latter three curves twice.

In fact k^{20} has a quadruple point in each of the points S_i, so that four of the points of intersection of LS_i with k^{20} lie in S_i, and the latter two curves count, therefore, for four of the curves k^3 through P_1 which touch φ; hence each of these curves must be counted twice. In order to

[1]) Cf. Prof. JAN DE VRIES, these Proceedings, 11, 84.

show this we make use of the fact that there are eight curves k^3 which pass through four given points, cut a straight line, cut another straight line twice, and touch a plane [1]). Among the 80 points of intersection of λ^4 and k^{20} the points L and M must each be counted 20 times. Further they contain the four points S_i', each counted twice, and the eight pairs of image points of the curves k^3 which touch φ and which cut l twice. There remain 16 points of intersection, so that four of the points of intersection of λ^4, and also of μ^4, with k^{20} coincide in each point S_i. And the curves k^{20} have no definite tangent in S_i as any k^3 may belong to Σ_2.

Accordingly the system Σ_2 of the curves k^3 which touch a plane φ, is represented on a curve of the order twenty, k^{20}, which has tenfold points in L and M, quadruple points in the points S_i and double points in the points S_i'.

A curve k^{10} and a curve k^{20} have sixty points, that are non-singular for the representation in common.

There are, therefore, sixty twisted cubics which pass through four given points, cut three given straight lines, and touch a given plane.

Two curves k^{20} have 120 points of intersection that are not singular for the representation.

Consequently there are 120 twisted cubics which pass through four given points, cut two given straight lines, and touch two given planes.

The sixteen non-singular points of intersection of λ^4 and k^{20} are the eight pairs of image points of the curves k^3 which cut l twice and which touch φ.

A curve k_i^{*4} has 26 points in common with k^{20} which are not singular for the representation.

The curves k^3 which touch a given plane, form a surface of the degree sixty, which has l and m as tenfold lines and has 34-fold points in the points H_i.

If φ passes through H_i, the surface of the curves k^3 which touch φ at H_i, discussed in § 4, splits off twice.

Accordingly the surface of the curves k^3 which touch a plane through H_i outside H_i, is of the degree 26. It has quadruple lines in l and m, an eighteenfold point in H_i, and fourteenfold points in the other points H.

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[1]) Cf. e. g. these Proceedings, **29**, 781.

Mathematics. — *"Determination of the System (2, 1, 1,) of ∞^3 Line Elements of Space"*. By Dr. G. SCHAAKE. (Communicated by Prof. HENDRIK DE VRIES).

(Communicated at the meeting of April 24, 1926).

§ 1. In my paper: *"Determination of the Bilinear System of ∞^3 Line Elements of Space"* [1]) the system (1, 1, 2) of ∞^3 line elements of space has been examined. In this paper we shall treat the system (2, 1, 1) of ∞^3 line elements (P, l) of space. For this system S_3 the order of the complex of the lines l is two; it contains one line element for which P lies in an arbitrary given point of space and the points P of the line elements of S_3 which belong to an arbitrary given plane, form a straight line r.

§ 2. The lines l which define line elements of S_3 with the points P of a given line m, form a scroll which has m as a single directrix. A plane through m contains one generatrix of this scroll, the line which, as line l, corresponds to the point P of m which lies in the point of intersection of the line r of the said plane with m.

The lines l which correspond to the points P of a line m, form, therefore, a quadratic scroll μ^2.

The locus of the points P of the lines of a complex cone $\varkappa$ of the quadratic complex C^2 of the lines l of S_3, is a curve which cuts a plane through the vertex T of $\varkappa$ in two points outside T, the points P of the two generatrices l of $\varkappa$ in this plane. Besides this curve passes through T where it touches the line l which forms a line element of S_3 with T.

Accordingly the locus of the points P of the generatrices l of a complex cone $\varkappa$ of C^2 is a twisted cubic which passes through the vertex of $\varkappa$.

The lines l of C^2 which cut two given lines m and n, form a biquadratic scroll λ^4 of the genus one, which has m and n as double directrices. The locus of the points P of the lines l of λ^4 cuts a plane through m outside m in the points P of the two generatrices l of λ^4 in this plane. There are two lines l of λ^4 for which the corresponding point P lies in m; they are the two generatrices of the scroll μ^2 corresponding to m which cut n.

Consequently the points P of the lines l of C^2 which cut the lines m and n, form a biquadratic twisted curve k^4 of the genus one which has m and n as chords.

The lines l of C^2 which cut a line m, form a congruence $M(2, 2)$. The locus of the points P of the lines l of a congruence $M(2, 2)$ is a

[1]) These Proceedings, **27**, 2.

surface which contains m, because among the lines which pass through a point of m, there is one which has its point P in the said point. To the points P where this surface cuts an arbitrary line m', there correspond as lines l the generatrices which cut m of the scroll μ'^2 corresponding to m'.

Accordingly to a congruence $M(2, 2)$ of lines of C^2 which cut a given line m, there corresponds a quadratic surface Ω^2 of points P which contains m.

§ 3. A curve of singular points P would lie on all surfaces Ω^2, as one or more of the ∞^1 lines l associated to a point of this curve, would cut an arbitrary line m. And the intersection of two surfaces Ω^2 is formed by the curve k^4 of the points P of the lines l of C^2 which cut the two lines to which the said surfaces Ω^2 correspond.

Accordingly the system S_3 does not contain any curve of singular points P.

Two of the six points of intersection of the curve k^3 associated to a complex cone $\varkappa$ and the quadratic surface corresponding to a line m, lie in the points P of the lines l of $\varkappa$ which cut m. The other four, to which there must correspond a straight line of $\varkappa$ as well as a line outside $\varkappa$ cutting m, are necessarily singular. As S_3 does not contain any singular curve, any curve k^3 and any surface Ω^2 have the same four singular points in common. Through these four points $H_1, \ldots, H_4$ there pass any surface Ω^2, any curve k^3 and consequently any complex cone $\varkappa$. Accordingly the points $H_1, \ldots, H_4$ are cardinal points of the complex C^2 and also of S_3, as each of these points corresponds as point P to any line l through it.

In a plane through three of the points H there lie three pencils of lines l of C^2, which, therefore, form a curve of the third class. As the lines of C^2 in an arbitrary plane envelop a conic, all the lines of a plane through three points H must belong to C^2 so that the sides of the tetrahedron $H_1 H_2 H_3 H_4$ are cardinal planes of C^2. The quadratic complexes through the congruence (4, 4) consisting of the four sheaves $H_1, \ldots, H_4$ and the four fields in the sides of the tetrahedron $H_1 H_2 H_3 H_4$, form a pencil of which one individual also contains a line l given at random. Consequently C^2 must be the tetrahedral complex of which $H_1 H_2 H_3 H_4$ is the tetrahedron of singularities and which contains a line of C^2 that does not pass through any of the angular points of the said tetrahedron and does not lie in any of the sides either.

Accordingly the complex C^2 of the lines of S_3 is a tetrahedral complex. The four cardinal points of this complex are also cardinal points of S_3.

§ 4. Of a complex cone $\varkappa$ of C^2 with vertex T we consider two generatrices l_1 and l_2 to which resp. the points P_1 and P_2 correspond in S_3. The curve k^3 of the points P of the lines of $\varkappa$ passes through

$T, P_1, P_2, H_1, H_2, H_3$ and H_4. If we consider the projective correspondence between the sheaves P_1 and P_2 in which to the lines $P_1 H_1, \ldots,$ $P_1 H_4$ there correspond resp. the lines $P_2 H_1, \ldots, P_2 H_4$, k^3 is the locus of the points of intersection of lines associated to each other in this correspondence. To the line l_1 through P_1 the line l_2 through P_2 is conjugated as these cut each other in the point T of k^3. This also holds good if P_2 lies for instance in T. In this case to the line $l_1 \equiv P_1 T$, the tangent to k^3 at T is associated which line l_2 corresponds to the point P_2 in T.

Let us now choose two points P_1 and P_2 so that the lines l_1 and l_2 do not cut each other. If P describes the line l_1, l describes a scroll with l_1 not only as directrix but also as generatrix, i. e. the system of tangents to a complex conic of C^2 in a plane through l_1, to which plane, accordingly, l_1 is associated as line r. Through the point of intersection of l_2 and this plane we draw a tangent l_3 to the said complex conic. To l_3 there corresponds in S_3 the point of intersection P_3 of l_3 and l_1.

In the projective correspondence between the sheaves P_1 and P_3 which is defined by the four pairs of rays $(P_1 H_i, P_3 H_i)$, the line l_3 corresponds to l_1 according to the above. In the same way the line l_2 corresponds to l_3 in the projective correspondence between the sheaves P_3 and P_2 which is defined by the four pairs $(P_3 H_i, P_2 H_i)$. Hence in the projective correspondence between the sheaves P_1 and P_2 which is defined by the four pairs $(P_1 H_i, P_2 H_i)$, the line l_2 corresponds to l_1.

Accordingly a system S_3 (2, 1, 1) of ∞^3 line elements may always be produced by the aid of four given points $H_1, \ldots, H_4$ and a given line element (P_1, l_1) by associating to each point P of space the line l through P which in the projective correspondence between the sheaves P_1 and P defined by the four pairs $(P_1 H_i, P H_i)$, corresponds to l_1.

Instead of a line element (P_1, l_1) of S_3 we may also choose five rays $s_1, \ldots, s_5$ which pass through an arbitrary point S, and we may associate to any point P the line l which in the projective correspondence between the sheaves S and P defined by the four pairs (s_i, PH_i) corresponds to s_5.

§ 5. On the line l_1 of the line element $(P_1 l_1)$ belonging to S_3, we choose a point Q_1 different from P_1. We consider the collinear transformation (P,Q) which has its coincidences in the points $H_1, \ldots, H_4$, and in which the point Q_1 corresponds to P_1. The join l of the points P and Q corresponding to each other in this transformation, form a tetrahedral complex which has its cardinal points in $H_1, \ldots, H_4$ and contains l_1, and which is, therefore, identical with C^2. If to any line l of C^2 we associate the point P of the pair (P, Q) on l, there arises a system (2, 1, 1) of ∞^3 line elements which has the cardinal points $H_1, \ldots, H_4$ and the line element $(P_1 l_1)$ in common with S_3 and which is, therefore, identical with S_3.

A system S_3 (2, 1, 1) of ∞^3 line elements can always be derived from a collinear transformation (P, Q) of space if to any point P the join l of P and the point Q corresponding to P in the transformation, is associated.

This leads to the well known theorem [1]), that the quintets of rays which join the points P of space with the coincidences of a collinear correspondence and with the points Q corresponding to P, can be transformed into each other by collinear transformations.

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[1]) STURM, Liniengeometrie, I. p. 354.

Mathematics. — *"The Involution in the Rays of Space with Rank Numbers One, One, and Zero"*. By Dr. G. SCHAAKE. (Communicated by Prof. HENDRIK DE VRIES.)

(Communicated at the meeting of May 29, 1926).

§ 1. On p. 19 of Vol. 28. of these Proceedings we have indicated how the involution of rays with rank numbers *one, one,* and *zero* is defined through two given straight lines m and n and a bilinear congruence K. For by the aid of these data we can construct the line l' associated to a line l by determining the line different from l which the scroll (l, m, n) has in common with the linear complex containing l and K.

By means of a representation of the rays of space on the points of a linear four-dimensional space this involution has been treated by the author on p. 84—92 of his thesis for the doctorate [1]).

In this paper we shall first investigate the said involution of rays without using more-dimensional considerations. Further we shall indicate the special involutions of rays with rank numbers one, one, and zero for which the scroll associated to a plane pencil and the congruence corresponding to a star and a field, have lower degree numbers than those for the general involutions of rays with the same rank numbers.

§ 2. If l is chosen in m, the linear complex through l and K, i.e. the linear complex C_m through m and K, is entirely defined; for the scroll (l, m, n), however, we may choose any scroll which contains m and n, so that any line of C_m corresponds as line l' to the line l coinciding with m. In the same way any line of the linear complex C_n through n and K corresponds to n.

Accordingly the lines m and n are singular rays of the third rank [2]). *To each of these lines there corresponds a linear complex which contains K.*

If, however, we choose l in K, there is one scroll (l, m, n) whereas there are an infinite number of complexes (l, K). In this case, therefore, any line of the scroll (l, m, n) corresponds to l as line l'. The ∞^2 scrolls (l, m, n) through the rays l of K form a complex. We shall try to find the number of lines of this complex which lie in a given plane pencil w. $m, n,$ and w define a bilinear congruence of which the directrices are

[1]) *„Afbeeldingen van figuren op de punten eener lineaire ruimte"*, Groningen 1922.

[2]) This means that to m as well as to n there correspond ∞^3 rays.

the transversals of m and n which lie in the plane of w and pass through the vertex of w. Any scroll (l, m, n) which contains a line of w, must lie entirely in the bilinear congruence (m, n, w), as m, n and the generatrix in w cut the two directrices of the said congruence. Now the bilinear congruence (m, n, w) has two lines l in common with K and each of the corresponding scrolls (l, m, n) has a generatrix in w. Accordingly the scrolls (l, m, n) corresponding to the lines l of K, form a quadratic complex C^2 which contains K.

A straight line of C^2 through a point P of m must belong to a scroll (l, m, n) which is degenerate and which, accordingly, corresponds to a line l of K which cuts m or n. The lines of C^2 through a point P of m form, therefore, the plane pencil defined by m and the line of K through P, and another plane pencil that has P as vertex and of which the plane passes through m and the point of n where the line of K in the plane (P, n) cuts the line n. Hence m and n are double lines of C^2.

The lines of K are, therefore, singular. To any ray l of K there correspond the generatrices of the scroll (l, m, n). These scrolls form a quadratic complex C^2 which contains K and of which m and n are double lines.

A line l is also singular if the corresponding scroll (l, m, n) has an infinite number of rays in common with the complex (K, l). As there exists no linear complex which contains K, m and n, for such a line l the scroll (l, m, n) must degenerate so that l must cut m or n.

If we choose a line l which cuts m, the scroll (l, m, n) degenerates into the plane pencil (l, m) and the plane pencil which contains n and has a ray in common with (l, m). Now, if l cuts m but does not cut n, the plane pencil which contains n, cannot lie in (K, l). For if this were the case (K, l) would contain two rays of the plane pencil (l, m), hence also m, so that m, n and K would lie in a linear complex. The entire plane pencil (l, m) lies in (K, l), if l is a generatrix of one of the plane pencils of C_m which contain m, i.e. the plane pencils defined by m and a line of K cutting m. These plane pencils form a special bilinear congruence K_m which belongs to the complex C^2.

There are two special bilinear congruences K_m and K_n of singular rays, which are composed of singular plane pencils, i.e. plane pencils of which any two generatrices are always associated to each other in the involution under consideration.

If a straight line l describes a pencil (F, φ) with vertex F in a plane φ, the scroll (l, m, n) always belongs to the bilinear congruence of which the transversal f_1 through F and the transversal f_2 in φ of m and n are directrices. If we associate to each other those generatrices of a plane pencil of this congruence through which there pass a scroll (l, m, n) and a complex (K, l) corresponding to the same line l of the plane pencil (F, φ), there arises a (1,1) correspondence in the former plane pencil. Accordingly the scroll consisting of the rays each of which is common

to a scroll (l, m, n) and a complex (K, l) which contain the same line l of (F, φ), has f_1 and f_2 as double directrices and is, therefore, of the fourth degree. The plane pencil (F, φ) belongs to this surface.

Consequently the lines l' which are associated to the generatrices l of (F, φ), form a cubic scroll ϱ^3 which has f_1 as a double directrix, f_2 as a single directrix. To the generatrices of (F, φ) in C^2 there correspond two lines of K, to the generatrices which lie in C_m and C_n are resp. associated m and n.

Accordingly to a plane pencil there corresponds a cubic scroll which contains m and n and which has two lines in common with the bilinear congruence K.

As F is a point of the double directrix f_1 of ϱ^3, two generatrices of this surface that lie in the plane pencil (F, φ), pass through F. These are double rays of our involution.

The double lines form a quadratic complex. This complex contains the congruences K, K_m and K_n, hence also the lines m and n. The lines of the complex through a point of m or of n form the plane pencil of K_m or K_n which has this point as vertex, and a pencil in a plane through resp. n or m. Hence the bilinear congruence of which m and n are the directrices, belongs to the complex of the double lines.

If for l we choose a directrix of K, (K, l) becomes the special linear complex that has l as axis, and l' coincides, therefore, with l. The double lines of the involution through a point of a directrix of K form two plane pencils. The plane of one of them passes through the other directrix of K, the plane of the other one touches the quadratic surface which is defined by m, n, and the chosen directrix. For if we choose l in the indicated second plane pencil, (K, l) is the special linear complex which has the chosen directrix as axis, and this directrix touches the surface (l, m, n) so that l' coincides with l. In the same way it is evident that the double lines in a plane through a directrix of K form two pencils one of which contains this directrix. Hence the two directrices of K are single lines of the complex of the double rays. Just as m and n they are lines of the surface of KUMMER of this complex.

If (F, φ) contains one of the lines m and n, for any generatrix l of (F, φ) the scroll (l, m, n) degenerates into (F, φ) and the plane pencil through n or m which has a line in common with (F, φ). In this case the latter plane pencil is associated to (F, φ).

If the plane pencil (F, φ) has a line l in common with K, the scroll (l, m, n) corresponding to this line l splits off from the associated ϱ^3. In this case, accordingly, there corresponds to (F, φ) a plane pencil which has a line in common with K, of which the plane passes through f_1, and of which the vertex lies on f_2. This plane pencil, associated to (F, φ), lies with (F, φ) in the linear complex through K that contains (F, φ), and it contains the lines which, together with the generatices of (F, φ), form pairs of associated directrices of that individual of the pencil of complexes

defined by the bilinear congruence (m, n) which is involutory with the complex through K and (F, φ).

If (F, φ) contains a generatrix of K_m or of K_n, a plane pencil of K_m or of K_n splits off from ϱ^3 and there remains a quadratic scroll which contains n or m and which has a line in common with K and with K_m or K_n. If (F, φ) contains a line of each of the congruences K_m and K_n, a similar plane pencil is associated to (F, φ). Especially a pencil with vertex on m or n and plane through n or m is transformed into itself.

The order of the congruence Σ associated to a star S of rays l, is equal to the number of pairs of rays (l, l') of which one line passes through S, the other one through a point T chosen arbitrarily. Such a pair of rays must belong to one of the scrolls containing m and n which pass through S and T and of which the transversals s and t of m and n through S and T are, therefore, directrices. Hence the line l through S of such a pair of rays must belong to the pencil which has S as vertex and which lies in a plane through t. t is a single directrix of the ϱ^3 corresponding to this plane pencil, so that one directrix of this scroll passes through T. Accordingly the order of Σ is one.

The pairs (l, l') of which l passes through S and l' lies in a given plane π, must belong to the bilinear congruence which has s and the transversal p of m and n in π as directrices. To the pencil that has S as vertex and of which the plane passes through p, there corresponds a ϱ^3 that has p as single directrix so that two generatrices of this surface lie in π. The class of Σ is, therefore, two.

The singular line of Σ is the transversal s of m and n through S. The congruence Σ contains m and n, which lines correspond to the rays of C_m and C_n through S, and with each of the congruences K_m and K_n it has the plane pencil in common to which the generatrix through S of K_m or K_n belongs. To the quadratic complex cone of C^2 that has S as vertex, there corresponds the scroll which is formed by the lines of K that cut s.

To a star of rays l a congruence Σ (1,2) is associated which contains m and n, which has a scroll in common with K and a plane pencil in common with K_m and K_n.

The rays of Σ through S form the quadratic cone of the double rays through S. Accordingly Σ has a singular conic. This is one of the conics that lie on the quadratic surface which is formed by the generatrices of K that cut s and that pass through the points of intersection outside s of this quadratic surface with m an n, i.e. the vertices of the plane pencils of K_m and K_n in Σ.

If S lies on m or n, for any line l of the star the scroll (l, m, n) degenerates into the plane pencil (l, m) or (l, n) and a pencil containing n or m which lies in the plane (n, S) or (m, S). Accordingly to a star with vertex S on m or n the fields of rays (n, S) or (m, S) are resp. associated.

If S lies on one of the directrices of K, the complex (K, l) for a line l outside K of the star is always the special linear complex which has the directrix containing S as axis. The line l' corresponding to l always lies in the plane through this directrix and the transversal through S of m and n. Accordingly also in this case a field of rays corresponds to S. Therefore, from the congruence (1,2) which generally corresponds to S, the bilinear congruence has split off which is formed by the scrolls (l, m, n) of the generatrices l of the plane pencil which S has in common with K.

As there are two pairs (l, l') of which one line lies in a given plane φ, the other one passes through a given point T, the congruence Φ associated to a field of rays φ has the order *two*. If we have a line l' in a plane π corresponding to a line l of φ, l and l' both cut the transversals f and p of m and n in φ and π. In this case the line l belongs to the pencil in φ which has the point of intersection of p with φ as vertex. p is a double directrix of the cubic scroll ϱ^3 associated to this plane pencil, so that one generatrix of ϱ^3 lies in π. The class of Φ is, therefore, one. With respect to m, n, K, K_m and K_n, apparently Φ entirely behaves as a congruence Σ. The scroll which Φ has in common with K, consists of the lines of K which cut the singular line of Φ, i.e. f.

To a field of rays there corresponds a congruence (2,1) *which contains m and n, which has a scroll in common with K, and a plane pencil in common with K_m and K_n.*

To a field of rays φ containing m or n the star is associated which has the point of intersection of φ with n or m as vertex. If one of the directrices of K lies in φ, there corresponds to φ the star of which the vertex lies in the intersection of the transversal of m and n in φ with the directrix of K in φ. To a bilinear congruence, which, according to the theorem of HALPHEN, has three rays in common with a congruence Σ as well as with a congruence Φ, a congruence Ψ (3,3) is, therefore, associated. A star and a field of rays wich contain m, to which resp. a field and a star containing n are associated, each contain, therefore, only one line of Ψ which is non-singular for the involution of rays, so that m, and also n, is a double line of Ψ. Any plane pencil of K contains two lines of Ψ, to wit those which are associated to the lines which the bilinear congruence corresponding to such a plane pencil, has in common with the chosen bilinear congruence.

Accordingly to a bilinear congruence there corresponds a congruence (3,3) *which has double lines in m and n and which has a scroll of the fourth degree in common with K.* This congruence (3,3) has two plane pencils in common with K_m as well as with K_n.

If the bilinear congruence B contains one of the lines m and n, for instance m, it has two lines, non-singular for the involution, in common with each of the congruences which correspond to a sheaf and to a field. In this case B corresponds, therefore, to a congruence $\Psi(2,2)$, which has a single line in m, a double line in n, as a star and

a field containing n have one non-singular ray in common with B, a star and a field containing m none. The bilinear congruence corresponding to a plane pencil of K, contains only one non-singular line of B, so that $\Psi(2,2)$ has a quadratic scroll in common with K.

In the same way it appears that a bilinear congruence containing m and n corresponds to a similar bilinear congruence. The same holds good for a bilinear congruence of which one of the directrices coincides with a directrix of K.

As a linear complex Γ has three rays in common with the ϱ^3 which is associated to a plane pencil, a cubic complex Γ' corresponds to Γ. This complex contains K_m and K_n, because Γ has one line in each singular plane pencil of the involution. The lines of Γ' through a point of m form in the first place the plane pencil of K_m that contains m and that has this point as vertex. The other lines correspond to the generatrices of the pencil of Γ in a plane through n. This plane pencil has a line in common with K_n so that a quadratic cone containing m corresponds to it. Hence m and also n is a double line of Γ'. As $\overline{\Gamma}$ has two lines in common with each scroll (l, m, n) corresponding to a line l of K, any ray of K is a double line of Γ'.

To a linear complex there corresponds a cubic complex which has m and n as double lines just as the generatrices of K, and which contains K_m and K_n.

If m is a line of Γ, the complex C_m splits off from Γ' and there remains, therefore, a quadratic complex which has n as double line and of which m and the generatrices of K and K_n are single lines. The quadratic complex which corresponds to a linear complex through n, has m as double line and contains n and the generatrices of K and K_m as single lines. A linear complex containing m and n, passes into a similar complex. The special linear complexes that have m and n as axes, are associated to each other. A linear complex which contains K, is transformed into itself. [1])

§ 3. We shall now investigate the special cases of the involution of rays which arise for special relative situations of m, n, and K.

In the first place we shall suppose that m, n, and K belong to a linear complex A. In this case the directrices of K also lie in a linear complex with the bilinear congruence (m, n) which has m and n as directrices. For the linear complexes which contain the congruence (m, n) and form a pencil, are all involutory relative to A, and the complex B of this pencil which contains one directrix of K, must also contain the other directrix associated to it relative to A.

If we choose a line l outside A, the line which is associated to l relative to B must cut all the generatrices of the scroll of which l, m,

[1]) For further properties of the involution of rays cf. the paper mentioned in § 1.

and n are directrices and must, therefore, belong to the scroll (l, m, n) but also to the complex (K, l), as (K, l) and B are involutory. In this case l' is the line associated to l relative to B. If l lies in A, $(l, m. n)$ belongs entirely to A and (K, l) coincides with A.

Accordingly if m, n and K lie in one linear complex, the involution of rays degenerates into that of the pairs of associated directrices relative to a linear complex and into the system of ∞^4 pairs of rays of the scrolls of A which contain m and n, hence into an involution with rank numbers one, zero, and zero and an involution with rank numbers zero, one, and zero.

We shall now consider the case that m coincides with one of the directrices of K. For a line l which cuts m, the complex (K, l) is the special linear complex that has m as axis, and the scroll (l, m, n) degenerates into two plane pencils one of which, (l, m), belongs entirely to (K, l), so that to a line which cuts m, there corresponds a plane pencil. Hence from the involution of rays there splits off the system of the ∞^4 pairs of rays which lie in plane pencils containing m. In the remaining involution of rays a scroll ϱ^2 corresponds to a plane pencil w, as from the ϱ^3 which in the general case corresponds to a plane pencil, the plane pencil containing m splits off which has a generatrix in common with w.

Now from the quadratic complex of the double rays which contains m, n and K, there splits off the special linear complex that has m as axis, so that there remains a linear complex C. This complex is defined by K and n.

The pairs of rays which the linear complexes through K have in common with a definite scroll that contains m and n, form an involution which has as double rays the generatrix o of this scroll different from n and belonging to C, and the line m. For if we choose l in m, (K, l) is the special linear complex which has m as axis, and the common rays of (K, l) and the scroll which contains m and n, coincide in m.

As a special case of the involution of rays with rank numbers one, one, and zero we get, therefore, the involution for which the line l' corresponding to l, is constructed in the following way. Let a linear complex C be given, a line m which does not belong to C, and a line n in C. The scroll ϱ defined by l, m, and n has a line o in common with C besides n. Now to l the generatrix l' of ϱ is associated which is harmonically separated from l by m and o.

This involution of rays has been treated by the author in these Proceedings, Vol. 28, p. 556.

If m and n coincide with the directrices of K, we get an involution of rays in which a plane pencil w' is associated to a plane pencil w. In this case the involution of the pairs of rays which the linear complexes through K have in common with a scroll containing m and n, has m and n as double lines.

Consequently to the involutions of rays with rank numbers one, one, and zero, the involution of the pairs of lines is associated which correspond to each other in a scrolled involutory collineation of space.

It it easily seen that these two involutions of rays satisfy the formulas of § 3 of the paper mentioned in § 1 of this communication.

§ 4. We have another special case if K is degenerate in a star T and a field τ which have a plane pencil (T, τ) in common, so that the vertex T of the star lies in the plane τ. In this case for any line l the complex (K, l) is the special linear complex which has as directrix the generatrix of the pencil (T, τ) which cuts l.

We have, accordingly, an involution of rays for which the line l' corresponding to a line l, is defined in the following way. The line l cuts one generatrix t of the plane pencil (T, τ). The line l' corresponding to l is the generatrix of the scroll (l, m, n) different from l which cuts t.

The scrolls (l, m, n) which correspond to the lines l through T, form the special linear complex R_1 which has the transversal r_1 of m and n through T as directrix. The scrolls (l, m, n) associated to the lines l of τ, form the special linear complex R_2 for which the directrix lies in the transversal r_2 of m and n in τ. The quadratic complex associated to K, has degenerated in this case into R_1 and R_2.

Now the complexes C_m and C_n are also special and have resp. the generatrix t_m cutting m and the line t_n of (T,τ) cutting n as directrices. The special bilinear congruence K_m has degenerated into the field (t_m, m) and the star (t_m, m), K_n into the field (t_n, n) and the star (t_n, n).

The cubic scroll ϱ^3 associated to a plane pencil, contains m, n, a generatrix in τ, and a line through T. To a plane pencil with vertex in the plane (t_m, m) there corresponds a quadratic scroll through n which has a generatrix in each of the planes (t_m, m) and τ. If the plane of the pencil passes through the point (t_m, m), there corresponds to it a quadratic scroll through n which passes through the points (t_m, m) and T.

The congruence Σ (1, 2) associated to a star S, has plane pencils in the star T and in the field τ which cut the transversal s of m and n through S just as all the other generatrices of Σ. Σ has also a plane pencil containing m in common with the star (t_m, m) and a plane pencil containing n with the star (t_n, n). The singular conic of Σ (1, 2) lies in τ and belongs to the pencil that has T and the points of intersection of τ with m, n, and s as base points. To the rays of a star with vertex S in τ the lines of the plane (r_1, s) through T are associated.

The congruence Φ (2, 1) corresponding to a field φ, has a plane pencil in the star T and in the field τ, and a plane pencil containing m or n in each of the fields (t_m, m) and (t_n, n). If the plane φ passes through T, a star with vertex in τ is associated to the field.

The congruence (3,3), which corresponds to a bilinear congruence and has double lines in m and n, has a singular point in T and a

singular plane in τ. Its generatrices through T form a quadratic cone, its lines in τ envelop a conic. The tangents to this conic in the plane pencil (T, τ) lie on the cone that has T as vertex. This congruence has also a plane pencil containing m in the star and in the field (t_m, m) and a plane pencil containing n in the star and also in the field (t_n, n).

The cubic complex associated to a linear complex which has m, n, the lines of the star T and those of the field τ as double lines, also contains the stars (t_m, m) and (t_n, n) and the fields (t_m, m) and (t_n, n).

Mathematics. — *"On the Bilinear System of ∞^3 Line Elements of Space"*. By Dr. G. SCHAAKE. (Communicated by Prof. HENDRIK DE VRIES).

(Communicated at the meeting of June 26, 1926).

§ 1. We consider two bilinear systems S_3 and S'_3 of ∞^3 line elements of space, i.e. systems of line elements (P,l) of which the carriers l form resp. the linear complex C and the linear complex C'; S_3 as well as S'_3 contain one line element of which the point P is given [1]).

The line elements (P,l) of which the carriers l lie in one plane pencil with the carriers of the line elements of S_3 and S'_3 of which the points lie in P, form a system Σ_4 of ∞^4 individuals.

The characteristic numbers μ and ν of such a system indicate resp. the degree of the cone of the carriers l of the line elements of the system for which P is a given point, and the number of points P which correspond to a given line l.

As the carriers l of the line elements of Σ_4 with common point P form a plane pencil, for this system $\mu = 1$. The carriers of the line elements of S_3 with point P on a given line l_1, form a cubic scroll ϱ^3 which has l_1 as single directrix, and the carriers of the line elements of S'_3 with point P on l_1 form a similar scroll ϱ'^3. If to a point Q of l_1 we associate the two points where l_1 is cut by the generatrices of ϱ'^3 in the plane through l_1 and the line of ϱ^3 through Q, there arises a (2,2)-correspondence on l_1. The four coincidences of this correspondence are the points P of l_1 to which there correspond in S_3 and S'_3 lines which belong to the same plane pencil as l_1. Hence the second characteristic number ν of Σ_4 is equal to *four*.

§ 2. In the communication indicated in the footnote we have derived that the singular points P of S_3, to each of which there corresponds a plane pencil of lines l, form a twisted curve k^5 of the order five and the genus one. The locus of the points P of the lines l of a bilinear congruence Γ of C consists of a cubic surface Ω^3 which contains the directrices r and r' of Γ and k^5.

The cubic surfaces Ω^3 through k^5 form a linear system of ∞^4 individuals. A surface Ω^3 contains two lines r and r' which do not cut k^5. For if we represent Ω^3 as simply as possible on a plane V, a cubic k^3 in V

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[1]) Cf. my communication: *Determination of the Bilinear System of ∞^3 Line Elements of Space*, these Proceedings, **27**, 2.

which passes through four of the six singular points, is the image of a quintic k^5 on Ω^3 of the genus one, so that only those two lines of Ω^3 do not cut the curve k^5 which are associated to the two singular points through which k^3 does not pass. These lines are the directrices of the bilinear congruence Γ of C of which the lines l have their image points on Ω^3. The point P which in S_3 corresponds to a line l of Γ, is the point of intersection outside r and r' of l with Ω^3.

From this we can derive a way to produce S_3. Let us consider a pencil Φ of surfaces Ω^3 through k^5. Besides k^5 this pencil has a base curve k^4 which is represented on a curve of the sixth order in V with two triple points and four double points and which is, accordingly, rational. The lines r and r' of the surface Ω^3 of Φ are always trisecants of k^4 and form, therefore, a scroll ϱ^2. The bilinear congruences of which the pairs (r, r') which define an involution on ϱ^2 are the pairs of directrices, form a linear complex, namely C.

The point P associated in S_3 to a line l of C, may be determined in the following way. We consider the surface Ω^3 of Φ which contains the two generatrices of ϱ^2 that cut l. Now the point in question P is the point of intersection outside these generatrices of l with the said surface Ω^3.

§ 3. Let us choose another quintic k'^5 of the genus one on an arbitrary surface Ω^3 through k^5 which, the same as k^5, does not cut the lines r and r' of Ω^3. In the way described in § 5 of the above mentioned communication or in the way just indicated we derive from k'^5 the bilinear system S'_3 of ∞^3 line elements. As a rule the linear complex C' which contains the lines l of S'_3, differs from C. We can make sure of this by choosing a trisecant of k'^5 outside C, as the surface of the trisecants of k'^5 belongs entirely to C', and in this case C' contains, accordingly, a line which does not lie in C.

For S_3 as well as for S'_3 Ω^3 is the locus of the points P of the lines of the bilinear congruence Γ which has the two lines r and r' as directrices. The scrolls ϱ^3 and ϱ'^3 that are formed by the carriers of the line elements of S_3 and S'_3 of which the points P lie on a given line l_1, have, therefore, always three common generatrices belonging to Γ. These generatrices cut l_1 in three of the four coincidences of the (2, 2)-correspondence on l_1 constructed in § 1. Consequently l_1 contains only one point P to which there correspond lines in S_3 and S'_3 that lie in a plane pencil with l_1.

Accordingly a bilinear system of ∞^3 line elements always consists of those line elements of a bilinear system of ∞^4 individuals of which the carriers lie in a linear complex.

§ 4. The loci of the points P of the line elements of S_3 and S'_3 that belong to a plane α, are resp. the conics k^2 and k'^2. One of the four

common points of these conics is the point which in S_3 as well as in S'_3 is associated to the line l of Γ which joins the two null points A and A' that correspond to α in C and C'. The other three points of intersection of k^2 and k'^2 are the vertices of the pencils in α that consist of lines l of the bilinear system Σ_4 of ∞^4 line elements which contains S_3 and S'_3. The joins of these three points of intersection, to each of which there correspond more than one point P in Σ_4, are, therefore, singular lines of Σ_4.

An arbitrary point P of space, to which in Σ_4 there corresponds a plane pencil (P, α) of lines l, must be one of the points of intersection of the conics k^2 and k'^2 in α which are different from the point P of the line AA'. Consequently two singular lines in α pass through P.

The lines l associated in Σ_4 to a point P of space, are, therefore, those lines which belong to one plane pencil with the two rays through P of a congruence $K(2, 3)$.

The line elements of Σ_4 which belong to a plane α, form a *bilinear null system* [1]). The singular lines of this null system are the rays of $K(2, 3)$ in α, the singular points are the angular points of the triangle which is formed by the singular lines.

As a congruence $K(2, 3)$ without singular curve has the rank one, and, accordingly, any line of space belongs to the same plane pencil with one pair of rays of K, inversely a bilinear system Σ_4 may always be derived from such a congruence. [2]). The singular lines of Σ_4 are the rays of $K(2, 3)$.

The five singular points of the second order of $K(2, 3)$ are evidently cardinal points of Σ_4 and singular points for S_3 and S'_3. They are, therefore, the five points of intersection of the curves k^5 and k'^5 from which resp. S_3 and S'_3 may be derived.

§ 5. The congruence $K(2, 2)$ of the lines which touch a quadratic cone $\varkappa$ on a give plane V, has the rank one. For if a point P describes a line r, the two lines of $K(2, 2)$ through P lie in one plane with r if the line of intersection of the polar plane of P relative to $\varkappa$ with V cuts the line r. This is only the case if P is the pole on r of the plane through the line associated to r relative to $\varkappa$ and the point of intersection of r with V.

The bilinear system Σ_4 of the line elements (P, l) of which the carriers l lie in one plane pencil with the rays through P of $K(2, 2)$, has evidently singular rays in the lines of V besides in the lines of $K(2, 2)$ so that also in this case the singular lines form a congruence (2, 3).

If for V we choose the plane at infinity and for $\varkappa$ an isotropous cone, in the bilinear system Σ_4 there corresponds to any line l the point of

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1). Prof. JAN DE VRIES: *Plane Linear Null Systems*, these Proceedings, 15, 1165.

2) Cf. STURM, *Liniengeometrie* II, p. 244.

intersection with the perpendicular which may be let down on l out of the vertex of the isotropous cone.

The system S_3 which consists of the line elements of Σ_4 of which the carriers belong to a given linear complex, has two curves of singular points, to wit the imaginary circle and a twisted cubic which cuts the imaginary circle twice and passes through the null point of the plane at infinity.

Another system Σ_4 associates to any line l the point of intersection P of l with the line which cuts l and a given line a at right angles. For this system the congruence (2, 3) of the singular rays consists of the lines of the plane at infinity, of the two isotropous planes through a, and of the star with vertex in the point at infinity of a counted twice. The star of rays must be counted twice because in this case the plane pencil of lines l associated in Σ_4 to an arbitrary point P, contains only one singular line which belongs to this star.

Mathematics. — *"A Special Congruence of Twisted Cubics"*. By Prof. JAN DE VRIES.

(Communicated at the meeting of May 29, 1926).

1. The twisted cubics ϱ^3 through the points A_1, A_2 and A_3 which have the straight line b as bisecant and the straight lines c_1, c_2 as secants, form a congruence $[\varrho^3]$. Let the plane of the points A be indicated by α, the transits of b, c_1, c_2 through α by B, C_1, C_2.

To this congruence there belongs the **pencil of nodal curves** k^3 which have a node in B and which pass through A_1, A_2, A_3, C_1, C_2. This pencil contains two cuspidal curves with cusp B and five figures consisting of a k^2 and a straight line.

2. In order to arrive at a *representation* of $[\varrho^3]$, I assume two pencils (p) and (q) in one plane with vertices L and M, and I establish a projective correspondence between the point range (P) of c_1 and (p), and likewise between (Q) of c_2 and (q). As the image of the curve through P and Q I consider the point $R \equiv p\,q$.

If P^* and Q^* are associated to the rays $L\,M$ and $M\,L$, the ϱ^3 defined by them is represented on the point range of LM; this curve is, therefore, *singular*.

All the curves through P^* have their image in M, all the curves through Q^* are represented in L. Hence L and M are *singular points* for the representation.

If p_0 and q_0 are associated to the points C_1 and C_2, their point of intersection S_0 is the image of any curve k^3 (§ 1). Accordingly also S_0 is *singular*.

Let a_1 be the plane through A_1 and b, A' the transit of a_1 $(A_2\,A_3)$, C'_1, (C'_2) the transit of c_1 (c_2). Each conic ϱ^2 through A_1, A', C'_1, C'_2 forms with a_1 a configuration belonging to $[\varrho^3]$; all these ϱ^3 have their images in the point of intersection S_1 of the rays p', q' corresponding to C'_1, C_2'. The planes a_2 and a_3 contain analogous systems. Also the points S_1, S_2 and S_3 are *singular*.

3. The surface Γ formed by the ϱ^3 which rest on a line c_3, is represented on a curve γ of which we shall determine the order.

Let P be a point of c_1. Through P and the points A there pass four ϱ^3 which cut b twice and which rest on c_2 and c_3. [1]).

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[1]) This well known number may be found in the following way by applying the principle of the conservation of the number. If the four points lie in a plane φ, in the first place the curve k^3 satisfies the conditions which has the transit of b as double point and which rests on the two secants. Further the 3 given lines define a hyperboloid which has three more points in common with the conic in φ passing through the given points and resting on b. With the straight lines of the scroll through the said points of intersection this k^2 forms three configurations ρ^3. Hence in the notation of SCHUBERT $P^4B\nu^2 = 4$.

Accordingly by Γ four points Q of c_2 are associated to P; hence a ray p cuts γ in four points outside L. In the same way four rays p are associated to the ray ML; hence γ has a quadruple point in L. It may, therefore, be indicated by the symbol γ^8 (L^4, M^4).

If we replace c_3 by a line c_3^*, hence γ by γ^*, the points of intersection of the two curves outside the singular points are the images of the curves ϱ^3 which rest on c_3 and on c_3^*. Accordingly through A_1, A_2, A_3 there pass 28 curves which have b as bisecant and which cut four given straight lines; $P^3 B \nu^4 = 28$. Consequently the *surface* Γ is of the *degree* 28.

4. The congruence contains still other composite configurations:

a. Any conic β^2 in α through A_1, A_2, A_3, B forms a ϱ^3 with any of the three lines r (PQ) which rest on β^2, b, c_1 and c_2. The lines r form a quadratic scroll $(r)^2$; to each r there corresponds one β^2. As (P) and (Q) are projective this system has as *image* the *conic* through the *singular* points L, M, S_1, S_2 and S_3.

b. The conic γ_1^2, $(A_1 A_2 A_3 B C_1)$ is completed to configurations ϱ^3 by any transversal r of b and c_2 which rests on it. These lines form a cubic scroll $(r)^3$ of which b is the double directrix. The system (γ_1^2, r) has as *image* the *point range* on p_0 (§ 2).

Analogously q_0 is the *image* of the system $(\gamma_2{}^2, r)$.

c. The conics $\delta_{1,1}^2$ through A_2 and A_3 which rest on b, on c_1, and on the transversal $t_{1,2}$ of b and c_2 through A_1, form a cubic dimonoid with double points A_2 and A_3. The line $t_{1,2}$ forms a ϱ^3 with any $\delta_{1,1}^2$. This system has as *image* the *point range* on the ray $q_{1,2} \equiv MS_1$, which corresponds to the point of intersection of c_2 and $t_{1,2}$.

There are five analogous systems; for we may interchange c_1 and c_2 and replace A_1 by A_2 or A_3.

The *images* of these systems are the *point ranges* on rays which may be indicated by $p_{1,1}, p_{2,1}, p_{3,1}$ and $q_{2,2}, q_{3,2}$.

d. The conics $\varepsilon_1{}^2$ through A_2 and A_3 which cut b, c_1 and c_2, form a dimonoid of the fourth degree with triple points A_2, A_3. To each $\varepsilon_1{}^2$ the transversal t_1 through A_1 of b and $\varepsilon_1{}^2$ is associated. Any ray of the plane pencil (t_1) corresponds to three conics. As again (P) and (Q) are projective, the system $(\varepsilon_1{}^2, t_1)$ has as *image* a *conic* through L, M and S_0.

Analogously there are the systems $(\varepsilon_2{}^2, t_2)$ and $(\varepsilon_3{}^2, t_3)$.

5. The degree of the surface Γ (§ 3) may also be determined by the aid of the intersection of Γ with α. This contains in the first place the curve k^3 which rests on c_3 (§ 1). Further the lines a_1, a_2, a_3 (§ 2), a conic β^2 (§ 4, *a*), which is triple, a $\gamma_1{}^2$ and a $\gamma_2{}^2$ (§ 4, *b*), which are likewise triple, and finally two conics β^2 of which the completing lines r rest on c_3. Apparently the complete intersection is of the order 28.

On Γ there also lie 18 conics δ^2 (§ 4, *c*), 21 conics ε^2 (§ 4, *d*) and 3 conics ϱ^2, which are completed by a_1, a_2, a_3 (§ 2). Further Γ contains the 9 straight lines corresponding to the above mentioned conics β^2, $\gamma_1{}^2$, $\gamma_2{}^2$

and the 2 lines r resting on c_3 (§ 4, a); also the 6 lines $t_{k,l}$ (§ 4, c), which are triple, 3 lines t_k (§ 4, d), likewise triple, and 12 single lines t_k.

The curves ϱ^3 which have c_1 as bisecant, lie on the quadratic surface defined by A_1, A_2, A_3, b and c_1. Each of the two points of intersection of this scroll with c_2, with each of the two points of intersection with c_3, defines a ϱ^3 which belongs to the congruence and which cuts c_1 twice. Hence Γ contains twelve cubic *nodal curves.*

The lines c are *quadruple* on Γ (§ 3).

The intersection of Γ with the plane α_1 (through A_1 and b) consists of a ϱ^2 (§ 2), the triple lines $t_{1,1}$ and $t_{1,2}$ (§ 4. c), the triple line t_1, which rests on c_3 (§ 5, d), four rays t_1, which are single, and the line b. Consequently the line b is *thirteen-fold* on Γ.

The consideration of the intersection of Γ with the plane α readily shows that A_1, A_2 and A_3 are *fourteen-fold* points.

6. Let A_4 be a point of c_1. The curves ϱ^3 through the four points A_k which cut b twice and which rest on c_2, form a surface of the 4^{th} degree (§ 3). The intersection of this surface with the plane through three of the points A_k consists of two conics; hence the points A_k are double points.

Consequently through A_4 there pass two curves of the $[\varrho^3]$ which cut a line g through A_1 outside A_1. If A_4 describes the line c_1, these two curves describe a surface which has a curve $\gamma^4 (L^2, M^2)$ as image. For any ray p contains two image points besides L, and the images of the ϱ^3 through Q^* (§ 2) lie in L. This surface contains two conics, which are represented in S_2 and S_3 (§ 2). $\gamma^8 (L^4, M^4)$ and $\gamma^4 (L^2, M^2)$ have 14 points in common besides the singular points L, M, S_2, S_3; accordingly the line g cuts Γ in 14 points besides A_1. This again shows that A_1, A_2 and A_3 are *fourteen-fold* points.

Botany. — *"Concerning the Function of Caffein in the metabolism of Paullinia cupana."* By Prof. TH. WEEVERS.

(Communicated at the meeting of February 27, 1926).

Among the various plants containing caffein Paullinia cupana is the one that has received least attention. This is due to the fact that the plant is little known, as it is found only in the basin of the Amazon and in large amount only at Maués, miles and miles from the coast. On a study-tour, which a subsidy from the Langerhuizen-Fund enabled me to undertake, I found at Para a number of well-developed plants.

Before entering upon a description of my experiments on this plant, I wish to acknowledge gratefully the helpful services received from the Brazilian authorities and the Dutch Embassy [1]).

The present paper deals only with the microchemical examination of the plant under consideration; the results of the quantitative macrochemical investigations will be made known in a future communication, together with those concerning Ilex paraguiensis St. Hil.

PECKOLT's [2]) experiments have demonstrated that the seeds of Paullinia cupana are rich in caffein (4½ % dry weight), while he maintains further that the leaves do not contain any caffein. In connection with my previous observations upon the plants containing xanthin-derivatives, viz. Thea and Coffea spec., Theobroma Cacao and Cola acuminata [3]) this statement seemed so strange to me, that it alone gave sufficient reasons to study also Paullinia. The method used has been described by H. BEHRENS in his "Anleitung zur mikrochemischen Analyse", and has also been applied by my wife and myself in the above investigation.

The parts of the plant are pounded with quick-lime and the dry powder is extracted with 96 % alcohol. After evaporating the alcoholic extract to dryness a portion of the residue is sublimated and from the form of the crystals and their extinction the presence of caffein in the cooled sublimate is determined. The quantity of the sublimate moistened by breathing affords an index to ascertain whether there is much or little caffein in the extracted parts of the plant.

Paullinia cupana [4]) belonging to the Sapindaceae is a liane, a tendril climber with uneven-pinnate leaves. Like many other plants of the tropical

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[1]) I also wish to express my thanks to Prof. Dr. E. J. LOHMAN, Sc. D., Dr. DUK, and KUHLMANN of Rio de Janeiro, as well as to Dr. P. LE COMTE, Sc. D., Para.

[2]) PECKOLT, Sitz. d. Wien. Akad. 1866; Ber. d. pharm. Ges. 1901.

[3]) TH. WEEVERS and C. J. WEEVERS—DE GRAAFF, Zitttngsverslag Kon. Akad. v. Wet. Amsterdam **11**, 342, 1902; **12**, 369, 1903.

TH. WEEVERS, Ann. Jard. bot. Buitenzorg 2e Ser. Vol. VI.

[4]) MARTIUS (Kast. Archiv. 1826, Lieb. Ann. 1840) found guaranin in Paullinia; BERTHEMOT and DECHASTELE Journ. Pharm. **26**, 1840 identified it with caffein.

rain-forest Paullinia always possesses at the same time leaves of various generations of buds that have developed with longer or shorter intervals of rest. The leaves overgrown with epiphytic lichens are old ones; those without lichens are recently full-grown.

Leaves.

a. very young leaves, coloured red by anthocyan, contain rather much caffein.

b. young, not quite developed green leaves: much caffein.

c. barely full-grown, green leaves: rather much caffein.

d. old, green leaves overgrown with lichen: no caffein.

e. yellow leaves, just falling off: no caffein.

In all cases the amount of caffein was compared per 1.5 grm fresh weight, not per equal number of leaves, so it cannot be deduced from these experiments when the increase of caffein stops and the decrease begins. Macro-quantitative examinations are needed to solve this question.

These experiments warrant the conclusion that the caffein first contained in the leaves disappears being either transported or taken up in the metabolism.

Petioles.

a. in very young petioles: rather much caffein;

b. in stalks of full grown and old leaves no caffein. Transport of caffein is therefore not very probable.

Shoots.

a. young, growing shoots: rather much caffein;

b. branches with incipient lignification and barely developed leaves: very little caffein;

c. in lignified branches with old leaves: wood and marrow appeared not to contain caffein; inner-, and outer-bark contained a small quantity; older ones traces in the cambium;

d. branches of 2 and 3 years: no caffein.

Roots.

In young seedlings caffein occurs in inner, and outer-bark, not in wood and marrow. Old roots were not examined.

Flowers.

Male flowers much caffein, perhaps traces of theobromin. (The flowers are too small to ascertain whether here, just as in Kola acuminata, the greatest amount occurs in the stamens.)

Female flowers much caffein, perhaps a little theobromin [1]).

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[1]) TSCHIRCH and GROGG, Handb. der Pharmakognosie, found theobromin in the seed-coat of the ripe seed.

The greatest amount of caffein is contained in the ovaries, especially in the ovulae; the wall of the ovary contains only very little of it. The growing fruits hold it as well in the ovula as in the arillus. With these young ovulae it was possible to perform a simple quantitative estimation of the caffein by evaporating to dryness the colourless alcoholic extract of 50 ovulae and by dissolving the rest in some c.c. of warm water. The aqueous solution was filtered off and finally shaken out. Chloroform was added to the solution. Then the mixture was poured into a separating funnel. All the caffein passed into solution in the chloroform, if this process was repeated some times. By evaporating the chloroform, to dryness a crystallizing residue was left, which was nearly pure caffein and was determined by weighing;

per 50 very young ovulae ± 50 mgrms, of which ± 10 mgrs in the arillus;

per 50 very young seeds (dry weight 4 grms) 255 mgrs, of caffein, that is 5.6 % or 4.5 mgs each.

The dry weight of one adult seed is ± 0.7 grms, the caffein ± 30 mgs.

We see, then, that caffein occurs in all young, developing parts of Paullinia, from which it disappears after the parts have attained full growth. It is only the seeds that behave differently, as the increase of the amount of caffein continues until the ripening. So this plant presents completely the same aspect of the occurrence of caffein as has been set forth previously for Coffea liberica. Whether the caffein is used during the germination has to be made out by further experimentation.

In Para, Paullinia cupana has periods of rest, in which no young shoots and sometimes no ripening fruits are to be seen; then, as appears from the above, no caffein occurs either in the leaves or in the branches, whereas shortly before a rather large quantity was present during the opening of the buds. With the falling fruits and seeds the plant looses caffein, the falling leaves however do not contain any caffein.

The conclusion at which I arrived previously by the study of Theobroma Cacao and Coffea liberica, viz that the xanthin derivatives can enter again into the metabolism, holds therefore with perfect justice for Paullinia cupana. Here too caffein is an intermediate product, not an end-product of the metabolism, a conclusion that may be confined more precisely by future quantitative investigation, but does not require this investigation to prove its correctness.

The question whether also in Paullinia the caffein is utilized for the generation of protein, and whether the final conclusion of previous investigations with tea, coffee, cacao and kolaplants is completely confirmed by Paullinia also in this respect, must be solved by the quantitative examinations.

Physics. — *"Further experiments with liquid helium. BA. Preliminary determinations of the latent heat of vaporization of liquid helium."* By L. I. DANA[1]) and H. KAMERLINGH ONNES. (Communication N°. 179c from the Physical Laboratory at Leiden).[2])

(Communicated at the meetings of June 27, and December 19, 1925).

The object of these experiments was to measure the latent heat of vaporization of liquid helium for pressures at and below one atmosphere by a simple method. Measurements of this thermal constant have not previously been made. They are of especial importance in the problems which present themselves at the lowest temperatures with liquid helium, as a check on the helium temperature scale, and for the purpose of correlation with the vapor pressure and density data by means of the CLAUSIUS-CLAPEYRON equation.

Description of the Method and Apparatus.

It was decided to adopt as simple a method as possible because the operation of the helium cryostat involves a sufficient number of difficulties without introducing extra ones. Fig. 2 shows the calorimeter set up in the helium cryostat, the latter being the same as that described previously in Comm. N°. 159 with the exception that in these experiments the helium flask was longer than usual. Fig. 1 shows the calorimeter itself drawn on a larger scale. The latter consists of a small DEWAR flask supported by an outlet tube C and containing an electric heating coil P. Through a small valve E the flask could be filled with liquid helium from the surrounding bath. The initial level of the helium was read off by a cathetometer from the millimeter marks engraved on the inner tube. After a current was passed through and the power measured, the final level was ascertained. From the volume and density of the liquid evaporated and the power input, the latent heat could be readily computed.

Before the sealing of the lower part of the calorimeter, the inner tube was carefully calibrated with mercury. Then followed the insertion of the heating coil and the sealing and evacuation of the flask. The heating coil, of 0.1 mm. constantan wire and of 100 ohms resistance, was wound in bifilar fashion on a mica frame. The current leads, of 0.2 mm. copper wire, led out through two side tubes F and F', Fig. 1, while the openings at the ends of the tube were closed with small brass caps soldered to

[1]) SHELDON fellow in physics, Harvard University, 1922—1923.

[2]) This paper was printed in an abridged form in this Academy's „Zittingsverslagen", 34, 1335, (1925).

the glass. It was permissible to connect the current leads outside the calorimeter since the resistances of those within were negligible in com-

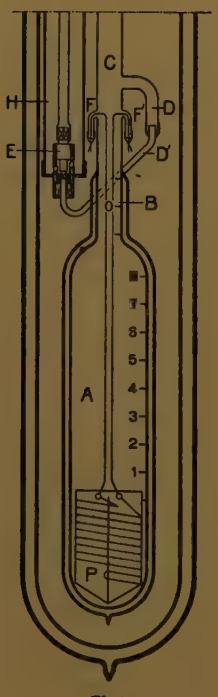

Fig. 1.

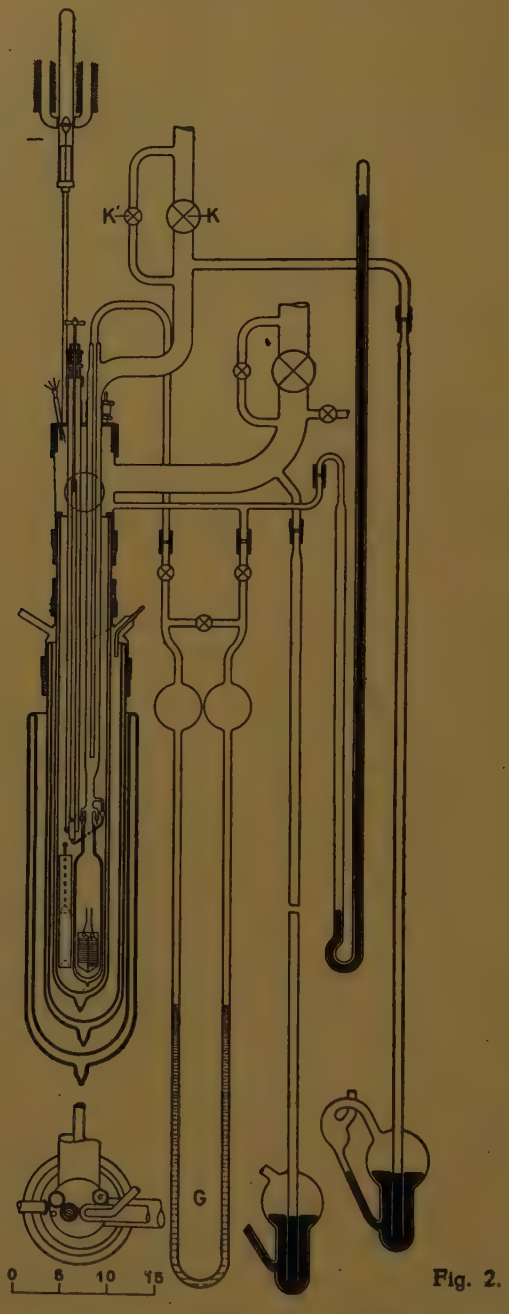

Fig. 2.

parison to that of the heating coil; this procedure dispensed with two extra outlet tubes.

The helium could have been condensed into the calorimeter under pressure, but as it was not allowed to have the level of the helium in the

cryostat above the part protected by the vacuum the time of filling would have been excessive. The use of the small valve E made it possible to fill the calorimeter quickly and thus to make a number of determinations on one day. A copper capillary D' made connection from the valve to the calorimeter, one end being soldered to the bottom of the valve and the other to the inlet tube D. To prevent the valve from rotating when the valve stem was turned, and the consequent breaking off of the glass tube D, the outer shell of the valve was soldered to the glass tube H, which reached to the head of the cryostat, and was there cemented in place. Steel is so hard at low temperatures that a steel valve stem would not make a tight fit with a steel seat; hence the seat was made of brass, while the stem consisted of hardened steel. A long handle, made of german silver tubing to reduce conduction of heat, fitting through a packing gland at the head of the cryostat, actuated the valve stem.

Initially, the diameter of the outlet tube was about 7 or 8 mm. It developed on trial that at the lower pressures of evaporation, below 10 mm. Hg or so, the frictional resistance to the flow of the helium at the rates of evaporation maintained was so great that with the lowest pressures which could easily be maintained with the pump at the top of the outlet, the pressure in the calorimeter was considerably higher. This necessitated a larger outlet tube (see Fig. 2) and, in addition, another concentric tube, leading to the top of the calorimeter, was installed as a static pressure measuring tube; thus no error as a result of a pressure drop could be introduced. The impact pressure of the helium flowing against the central tube was found to be negligible.

Obviously, to fill the calorimeter with helium a sufficient quantity had to be formed to cover the valve; further, it was desirable at all times to have a considerable height of the liquid to cut down heat leaks. It is evident that if the vapor pressure of the liquid inside the calorimeter is somewhat higher than that outside along the outlet tube, or what amounts to the same thing, if the temperature inside is slightly higher, then, while the vapors resulting from evaporation passed through the outlet tube, they would come in contact with the glass wall cooled by the colder liquid and would partly condense, the result being a large value of the observed latent heat. Thus previous to and during evaporation it was essential to keep the vapor pressure of the helium in the outside bath somewhat higher than inside the calorimeter in order that the vapors should be allowed to come in contact only with warmer spots than itself. This pressure difference was maintained by a valve K, Fig. 2, connected to the calorimeter and the vacuum line, and observed on a differential oil manometer G, one leg of which led from the outer bath and the other to the calorimeter. The differential height of the oil was so regulated as to give at the various pressures of evaporation a temperature difference of 0.05° C., — one which experiment showed to

suffice. To help maintain uniformity in temperature in the outer bath, a pump stirrer, shown in Fig. 2, was situated by the side of the calorimeter.

To read the levels of the helium meniscus, the lighting of the bath had to be arranged with care. The line of sight was tangent to the millimeter marks. Slits in the silvering of the liquid air and liquid hydrogen flasks, which surrounded the helium flask, limited the light passage; the lower part of the helium flask was unsilvered. Although the rays passed through eight glass walls, the distortion appeared to be small. A low wattage electric lamp behind the outer flask furnished the illumination and an alum cell between the two absorbed the infra-red rays. By inserting a sheet of black paper between the cell and slit, and by raising it till the edge stood just below the meniscus, the latter could be made to stand out sharply. This screen also served to protect most of the liquid in the calorimeter from the lamp rays, with the result that evaporation due to external radiation remained very small.

A WOLFF potentiometer of the DIESSELHORST type served to measure the current by determining the potential drop across a standard in series with the heating coil, and to measure the voltage across the heating coil by ascertaining the potential drop across a part of a standard resistance shunted across the heating coil. Proper corrections were always made for the current shunted through the standard resistance. The time of power input averaged 1000 seconds and to measure it a stop watch was employed.

Heat Leaks and other Sources of Error.

Perhaps the greatest difficulty to overcome in the study of the calorimetric properties of helium is the leakage of heat, because of the low density and the consequent small mass of liquid that can be put into the calorimeter, and principally because of the very low value of the latent heat, which is the smallest of any substance. The enclosure of the liquid to be measured in a small DEWAR flask helps greatly in the limiting of the leakage from the surroundings, but the necessity of having current leads and of maintaining a slightly higher temperature in the liquid outside introduces an unavoidable leak. The copper leads were chosen as small as consistent with mechanical strength. Further, other avenues of leakage are along the inner glass wall of the neck, and down the column of vapor existing in the latter. The amount of heat leaking along these two paths can be shown by calculation to be very small.

At any rate, the method of experiment was such as to take account of the leakage from the surroundings. Observations of level were taken for at least five minutes with no power input after the calorimeter had been filled and pressure equilibrium attained. When the evaporation of liquid by the current had ceased, readings of the leak were again taken. As was to be expected, the final leak resulted in general smaller than the initial one because of the longer path of conduction down the copper leads to the liquid. The correction to the volume evaporated by the

leak was assumed to be the mean of the initial and final values. It could be maintained that during evaporation the leakage was less than either before or after because of the absorption of some of it by the helium vapors flowing by the wires and the neck of the flask. However, the facts that the velocity of vapor did not reach high values, that the latent heat did not appear to be affected by the rate of evaporation, and that the leak on the average amounted to not more than 1 % of the total volume evaporated, appear to show that any error due to the assumption of the mean leak could not be greater than the error in any individual experimental factor.

Since the boiling of the helium under power input is accompanied by the formation of bubbles of vapor, it may be thought that the bursting of these bubbles results in the formation of a considerable quantity of mist and spray which is carried along with the evaporated vapor, the net result being a quantity of liquid removed without any corresponding heat input and a consequent low value of the observed latent heat. Nevertheless, the rate of evaporation was not so great as to cause violent ebullition, and if such effects did occur, it might be expected that they would vary greatly with the rate of evaporation; the results of the latent heat for various rates would appear to show that these effects are negligible.

Since the heating took place at the bottom of the liquid, the formation of bubbles was beneficial in so far as it produced stirring and better uniformity of temperature throughout the liquid. Further, any colder liquid at the top would give rise to convection currents, at least for temperatures higher than that of the maximum density of the liquid.

A height of 6 to 6.5 cm. of liquid usually was evaporated off. Thus the hydrostatic pressure on the lower levels of the liquid varied during evaporation. Because liquid helium is so light, the maximum effect due to this cause at the lowest evaporation pressure, 4 mm. Hg, is 0.3 mm. which is equivalent to about 0.03° K. and which is nearly negligible.

In ascertaining the levels of the liquid, the micrometer eyepiece of the cathetometer telescope served to determine the difference in height between the meniscus and the nearest millimeter mark on the inner wall of the calorimeter. Under the conditions of the experiments, with slight tremors of the helium meniscus due to vibration of compressors, with slight distortion of the meniscus, and with the necessity of taking readings rapidly, it is possible that there may be an error of 0.05 mm. in each meniscus reading, or say, an error of 0.1 mm. for both. The error in volume due to an error in meniscus height may be about 0.5 %; and this is probably the largest single experimental error. Both the power input and time measurement could readily be ascertained with an accuracy of 0.1 %.

Method of Operation.

When some liquid helium had been formed in the outer helium flask,

a slight vacuum was produced in the calorimeter, the valve opened, and a current of cold helium vapor passed through to cool off the inner walls of the calorimeter. When liquid covered the valve, it was allowed to flow in. The liquid stood for a few minutes with the proper pressure difference between the inside and outside in order to attain equilibrium and to make sure that any liquid remaining in the copper capillary D' had boiled off. It required but a short time for the latter to occur for the capillary rested in direct thermal contact with the warmer liquid outside. Then the heat leak observations were begun. Between the last reading of the liquid level and the beginning of the current a short time necessarily elapsed during which the level fell slightly. By noting this time interval with the stop watch the exact level at the time of the beginning of the power input was readily ascertained. At the end of the experiment the same procedure held but the correction was somewhat larger and of opposite sign; as a rule, it amounted to less than 0.1 % of the volume evaporated.

Results.

The results of the experiments are summarized in the table below together

Series	N°.	Pressure	Temp.	Latent heat of vaporization
A	I	764 mm.	4°.21 K.	4.95
	II	496	3.79	5.37
	III	153	2.96	5.64
B	I	768	4.21	4.89
	II	338	3.48	5.56
	III	10.4	1.81	5.55
	IV	25.9	2.14	5.58
	V	74.3	2.58	5.58
C	I	211	3.15	5.66
	II	108	2.77	5.60
	III	46.2	2.36	5.47
D	I	31.7	2.22	5.53
	II	4.26	1.49	5.45
	III	14.5	1.93	5.57
	IV	58.1	2.47	5.59
	V	9.04	1.76	5.52

with the important factors entering into the latent heat. The temperatures of the liquid, corresponding to the vapor pressures, were derived from the formula given in Comm. N°. 147*b*. Densities of the liquid are derived from recent measurements by KAMERLINGH ONNES and BOKS [1]). Because the same kind of glass was used in the construction of the calorimeter as in those experiments, the latent heat results are independent of the value assumed for the coefficient of expansion of the glass in the density measurements. The latent heats are given in 15° calories.

These results are plotted in Fig. 3.

Discussion of Results.

From the observations indicated by ⊙ it follows that the curve, representing the heat of vaporization as a function of the temperature shows a maximum near 3° K. and descends again at lower temperatures [2]). This should support the calculations of VERSCHAFFELT (c.f. Comm. Suppl. N°. 49) who arrives at the conclusion, that at $T = 0°$ K. the heat of vaporization

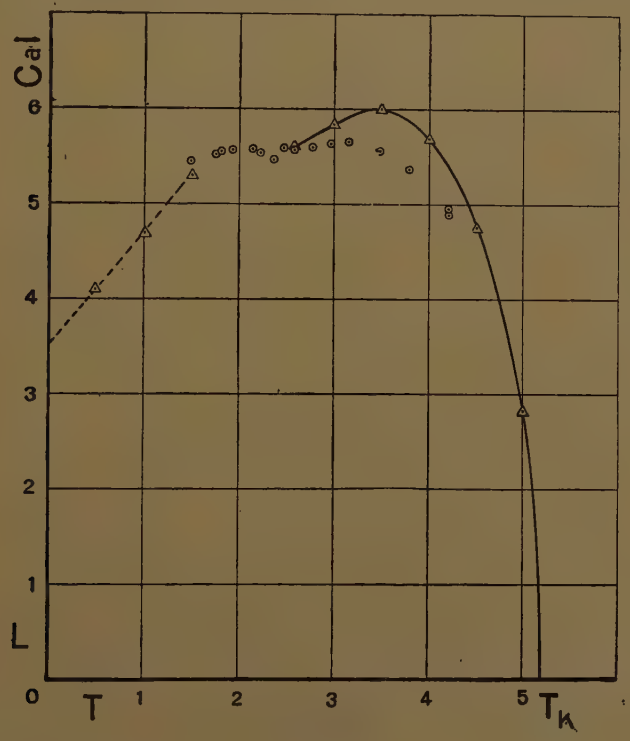

Fig. 3.

[1]) Leiden, Comm. No. 170, p. 22.

[2]) Further there is a noteworthy anomaly at the temperature of maximum density. It is possible that only the variation of the state of convection at surpassing this temperature causes a change in a systematic error. The accuracy of the results, however, being sufficient with regard to the deviation with near temperatures, it is remarkable that these results would indicate, that near the maximum density something happens to the helium, which within a small temperature range is perhaps even discontinuous. The change of density of the liquid indicates also something of the same kind.

of helium (solid or liquid) would be about 3.5 cal., and that near the absolute zero the heat of vaporization would change with temperature according to the formula:

$$L = 3.5 + 1.25\,T.$$

The curve, which represents graphically the formula (*b*) pg. 1059, has been drawn as a dotted line. This formula (*b*) agrees with the formula given above for small values of T. The joining of the curve of VERSCHAFFELT with the curve of the observations is still giving difficulties [1]).

Our observations, considering the small accuracy of the data for the vapor pressure, agree moderately well with the values L, which can be calculated from the data of MATHIAS, CROMMELIN, KAMERLINGH ONNES and SWALLOW by making use of the formula of CLAPEYRON-CLAUSIUS. One obtains then:

T	$v_{vap.}$	$v_{liq.}$	$\frac{dp}{dT}$ mm. Hg/degree	L
3° K.	233	6.9	270	± 5.8
3.5	127	7.3	450	6.0
4	77[5]	7.7	640	5.7
4.5	46[5]	8.4	870	4.0
5	26	10	1110	2.8

These points are also represented in the figure by △.

The values of $\frac{dp}{dT}$ were calculated from the formula

$$\log_{10} p = 2.19 - \frac{3.1}{T} + 2.5 \log_{10} T - 0.027\,T, \quad . \quad . \quad . \quad . \quad (1)$$

which was derived from the following reduced formula

$$\log_{10} \pi = -\frac{0.60}{\tau} + 2.5 \log_{10} \tau + 0.74 - 0.14\,\tau,$$

by using $T_k = 5.19°$ K. and $p_k = 2.26$ int. atm. or 1720 mm. Hg.

The reduced formula was obtained by adding to the formula of VERSCHAFFELT (loc. cit.) a correction term, so that for $T = T_k$ also $p = p_k$, thus for $\tau = 1$, $\pi = 1$. The formula thus obtained agreed only little less well with the observations than the formula in Leiden Comm. N°. 147*b*; one obtains namely:

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[1]) Within the limits considered indeed the temperatures, which can be calculated from this formula and from the formula of Comm. N°. 147*b* for a definite value of p, deviate from each other by several hundredths of a degree.

T	p mm. Hg.	p atm.	$\frac{dp}{dT}$ atm./degree
1.5	3.3	—	—
2.0	22	0.029	0.085
2.5	75	0.099	0.210
3.0	180	0.237	0.35
3.5	375	0.50	0.59
4.0	630	0.82	0.84
4.5	1030	1.32	1.15
5.0	1520	2.00	1.46

By using the formula:

$$\frac{d \log p}{dT} = \frac{ML}{RT^2} = \frac{4L}{2T^2} \quad \ldots\ldots\ldots \quad (a)$$

one derives from the vapor pressure formula in the neighbourhood of $T = 0°$ K.:

$$L = 3.5 + 1.25\,T - 0.03\,T^2 \quad \ldots\ldots \quad (b)$$

so that for

T	L
0.5	4.1
1.0	4.7
1.5	5.3

These points are also represented in the figure by $\triangle$.

At higher temperatures the simple formula (a) becomes too inaccurate because of the deviations from the gas laws, which the vapor shows. Even at $\pm 2.5°$ K. these deviations must still be considerable after the isotherm determinations of KAMERLINGH ONNES and BOKS. One has to apply then the correction, following from the observations of KAMERLINGH ONNES and BOKS, or when this is no longer satisfactory, one has to calculate at once with the formula of CLAPEYRON. The specific volume of the saturated vapor, as far as it has not been observed, can then approximately be calculated with the observations in question for some temperatures.

From the values of L obtained by making use of the formula of CLAPEYRON-CLAUSIUS, just as from our own observations it seems to follow that the heat of vaporization of helium reaches a maximum at

3.5° K., by which our previous surmise[1]) is established. Moreover the course of L, so calculated, seems to indicate that also at temperatures lower than 2.5° K. the simple formula (a) is not yet applicable, and that this formula is not valid until immediately near $T=0$; this should indicate that, also to lower than 2° K. thus at a reduced temperature lower than 0.4, the behaviour of the saturated vapor of helium deviates still appreciably from the ideal gas laws. However, it could also be possible, that the value of L_0 is another one than was adopted by VERSCHAFFELT.

[1]) H. KAMERLINGH ONNES. Rec. Trav. Chim. Pays-Bas (4) 3, 535, 1923.

Physics. — *"Further experiments with liquid helium. BB. Preliminary determinations of the specific heat of liquid helium."* By L. I. DANA [1]) and H. KAMERLINGH ONNES. (Communication No. 179*d* from the Physical Laboratory at Leiden). [2])

(Communicated at the meetings of June 27, and December 19, 1925).

Measurements on the specific heat of liquid helium have not previously been made. They are of great importance in the study of the liquid state near the absolute zero. The object of these experiments has been, not so much to measure the specific heat with high precision as to get a reasonably good idea of the values and their trend with the temperature for saturation pressures at and below one atmosphere as far as could be conveniently gone.

Description of the Method and Apparatus.

In Fig. 1 is shown the general set-up of the calorimeter in the helium cryostat. In some ways, the arrangement is similar to that of the latent heat apparatus. (Comm. N°. 179*c*). The specific heat calorimeter is drawn on a larger scale in Fig. 2. The body is a small DEWAR flask supported by the tube *B* which leads to the vapor pressure manometer. The calorimeter was filled with liquid helium from the outer bath through the small valve *A*.

To measure the specific heat arrangements had to be provided to (*a*) heat the liquid (*b*) stir the liquid (*c*) measure the rise in temperature.

The heating coil of manganin wire was wound on a grooved glass core *E* which had its support at the top of the inner wall by two glass stems. To stir the liquid and thus to maintain approximate uniformity of temperature especially during heating, a small glass plunger *F*, containing an iron wire was moved up and down by moving the iron ring *R* in the magnetic field produced by the magnet *M*, Fig. 1. This scheme for magnetic stirring is described in Leiden Comm. N°. 161.

Since the means of temperature measurement at these very low temperatures are limited, the simplest procedure to follow in measuring the rise in temperature of the liquid helium produced by heating appeared to be to determine the rise in vapor pressure; that is, to use the whole calorimeter as a vapor pressure thermometer. Especially with this method did the liquid have to be stirred, for the vapor pressure reading would be principally governed by the temperature of the meniscus. The mercury

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1) SHELDON fellow in physics, Harvard University, 1922—1923.

2) This paper was printed in an abridged form in this Academy's „Zittingsverslagen" **34**, 1340, (1925).

manometer adopted to measure the rise in pressure is shown in Fig. 1. This rise in pressure, before, during, and after the power input, had to

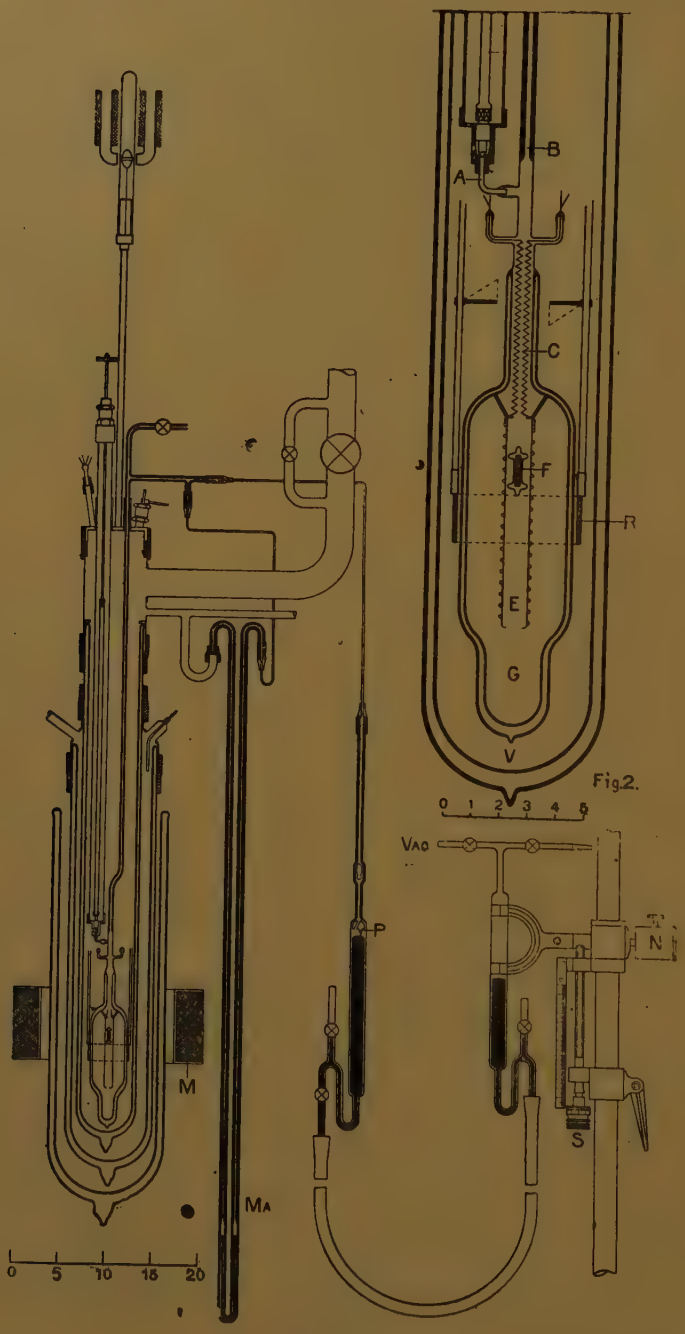

Fig. 1.

be determined quickly so that the arrangement shown in Fig. 1 of adjusting one meniscus to a fixed position and reading the position of the

other was used. A flexible rubber tube connection between the top and bottom of the manometer allowed freedom of motion of the upper part which was attached to a moveable nut N, actuated by a micrometer screw S. As the pressure rose, one observer maintained the lower meniscus at the fixed point P by sighting through a telescope and rotating the screw S while another observer recorded the readings of the screw. The actual pressure was read with a cathetometer while the rise in pressure was given by the screw reading; the latter being 11 cm. long, limited the rise in pressure for one experiment to about 10 cm. Hg.

The time of power input, measured with a stopwatch, averaged about 500 seconds. The current and voltage were measured with a calibrated standard WESTON voltmeter and ammeter; the power was of the order of 0.01 watt.

Before an experiment began, the liquid helium was brought to a level somewhat below the bottom of the neck of the calorimeter so that at the end of the power input the level should still be below. If the meniscus rose up high into the neck the temperature of the meniscus was very much influenced by the heat leaking in and false pressure readings would be obtained. After the experiments with a particular filling of the liquid were over, the liquid was warmed up until the meniscus rose into the neck. Since the calorimeter volume for various points in the neck had previously been determined, and the density of the liquid was known, the mass of the liquid filling could readily be ascertained. The bottom of the calorimeter was drawn down to facilitate the insertion of extra glass to determine the heat capacity of the calorimeter itself. But previous measurements on the specific heat of glass have indicated that the values are so small at helium temperatures that the correction for the heat capacity of the calorimeter would be very small and thus an extrapolated value could be assumed.

Heat Leaks.

It was demonstrated during the beginnings of the experiments, even though the specific heat determination did not involve the flow of matter out through the capillary, that if the outer bath were colder than the inner one, the possibility of condensation in the capillary leading to the manometer still existed. In the latter case, the vapor would condense, fall below, and new vapor would arise to go through the same process; this was a so-called convective heat leak, and when it did occur it produced a very large effect on the observed specific heats. To avoid this the same scheme as used in the latent heat experiments was adopted, namely, the maintenance of the outer helium bath at a somewhat higher temperature than the liquid in the calorimeter. In doing this with a differential manometer the essential condition of maintaining the volume of the dead vapor space as small as possible had to be observed, for toherwise the correction for evaporation of gas into this space would

have been too large. Thus, instead of an oil manometer, the mercury manometer *Ma*, Fig. 1, was employed. The difference in pressure had to be kept small to reduce the heat leak, and it was measured by a cathetometer. Due to the sudden input of current at the beginning of heating, it was difficult to maintain the desired pressure difference from the beginning, and it could happen that for a moment or so the surface of the outer bath became colder than the gas inside the capillary and condensation might result. A very small amount of condensation produces a very large error in the observed specific heat, for the heat input in the specific heat experiments is so much smaller than in the case of the latent heats. The effect was further increased because mercury instead of oil was used as the manometer fluid. As a result of this condensation a large number of points were erroneous. As was to be expected these points fell very irregularly off the curve obtained when condensation did not occur and always resulted in large values of the observed specific heat. Since the cause of this effect was well known from the latent heat experiments, where it could more easily be avoided and since the effect could be produced at will in the specific heat experiments, we felt justified in discarding the irregular points. The frequency of occurrence of condensation could be minimized by working with the level of the liquid in the outer bath somewhat below the top of the neck of the calorimeter. Measurements in this case could not proceed for a long period because the level would quickly fall to so low a value as to entail a large heat leak.

Vapor Space Correction.

It is evident that the exact volume of the vapor to which the measured heat input is applied is somewhat indefinite because the boundary of the calorimeter at the top is somewhat indefinite. Nevertheless, if we make the assumption that the saturated vapor space terminated at the bottom of the capillary tube, and assume that the vapor as well as the liquid took part in the rise in temperature, then it turns out that the volume of this vapor space need not be accurately known in determining the correction to be applied to the specific heat of the liquid. This correction is more dependent on the variation of the latent heat and specific volumes with the temperature; it is given by the following formula:

$$-\frac{L_2 x_2 - L_1 x_1}{\triangle T} + \left(\frac{Lx}{T}\right)_m$$

where L, x, T are respectively the latent heat, dryness factor, and absolute temperature, the subscripts 1, 2, and m referring to the lower, upper, and mean temperatures respectively.

The dryness factor is given by

$$x = \frac{\frac{V}{M} - u}{u' - u}, \text{ in which}$$

V is the total volume of the calorimeter,
M is the mass of liquid in the calorimeter,
u' is the specific volume of the vapor, and
u is the specific volume of the liquid.

This first correction as calculated for five points is given in the following table, from which the general manner of variation can be seen.

Mean T °K.	Correction Cal./g.
4.00	+ 0.057
3.79	+ 0.020
3.48	+ 0.007
2.99	— 0.003
2.62	— 0.014

These values are probably good to 10 %, entailing an error less than 1 % in the specific heat.

Finally, two other corrections are involved, that due to the heat capacity of the calorimeter, which amounts to about 1 % of the specific heat, and that due to evaporation into the superheated gas space, amounting to about 0.05 %.

Thus the equation giving the specific heat of the liquid in terms of the measured quantities is as follows:

$$C_{He} = \frac{Wxt}{4.184\, m_{He}\, \Delta T} - V.S. - \frac{c_g\, m_g}{m_{He}} - \frac{m_v\, L_m}{m_{He}\, \Delta T}$$

in which

W is the power input in watts,
t is the time of heating in seconds,
c_g and C_{He} are the specific heats of the glass and liquid helium respectively,
$V.S.$ is the vapor space correction given above,
m_{He}, m_g, m_v are the masses of the liquid helium, glass, and evaporated super-heated vapor, respectively,
L_m is the mean latent heat of the liquid helium, and
ΔT is the rise in temperature due to electric heating.

Procedure.

After filling the calorimeter with liquid and adjusting the temperatures of the outer and inner baths to the proper difference, readings were taken of the rise in pressure for about five minutes, and when plotted against the time gave a straight line. During the application of the heat,

the pressure was also read, and finally, the readings were continued for five minutes after the current ceased. For about 30 seconds after the latter ceased the pressure fell slightly and then proceeded to rise slowly. Evidently the rate of stirring was not sufficiently great to produce instantaneous uniformity of temperature during heating, but this lag was small and correction for it could readily be made. To get the initial pressure the leakage line was extrapolated to the instant of beginning of the current, while the final pressure was obtained by extrapolation to the instant of ending of the current.

Due to a variety of causes, the initial and final rates of leakage were not equal; but the change was very probably gradual. On the average, the temperature rise due to leakage was about 4 % of the total, and by subtracting the mean value from the total rise during power input, the correct rise was obtained.

To avoid errors due to calculations, to make the values conform to the temperature scale as given by the vapor pressure curve formula in Leiden Comm. N°. 147*b*, and to facilitate the reduction of the data, the temperature rise for each experiment was determined as follows. The values of $\frac{dp}{dT}$ as calculated from the above formula were plotted against the pressure p. The value of $\frac{dp}{dT}$ for the mean pressure of the experiment was read off, the reciprocal $\frac{dT}{dp}$ found, and this was multiplied by the corrected pressure rise to get the actual rise in temperature. It is proper to take the value of $\frac{dT}{dp}$ at the mean pressure provided it varies practically linearly with the pressure in the range of pressure under consideration. Over most of the range this is the case, but between about 20 and 150 mm. Hg, $\frac{dT}{dp}$ varies rapidly with the pressure. For this range $\frac{dT}{dp}$ was plotted against p and for any particular range $\Delta T = \int_{p_1}^{p_2} \frac{dT}{dp}\, dp$ was determined by graphical integration.

Results.

In the following table is given a summary of the results and the factors involved. It will be noted that the temperature scale used is that given by the vapor pressure formula in Leiden Comm. N°. 147*b*. The specific heats C are those corresponding to saturation conditions. Considering the deviations of the results, the experimental conditions, and the corrections we should say that the order of accuracy of the results should be about 3 %.

The specific heat of liquid helium.

Series N°.	Run N°.	Temp. Rise, °K. degr.	Pressure Rise mm. Hg	Power Input Watts	Mass of Liquid	Mean Pressure mm. Hg.	Mean Temp. ° K.	Specific Heat Cal./g.
I	I	0.103	69.3	0.00644	7.768	621	4.00	0.963
June 29	II	0.121	74.2	0.00665	„	523	3.84	0.919
1923	III	0.1065	70.9	0.00604	„	603	3.97	0.974
	IV	0.205	48.2	0.00604	8.076	99	2.72	0.602
	V	0.221	72.0	0.00927	„	180	3.05	0.717
	VI	0.455	70.5	0.02862	„	78	2.61	0.644
	VII	0.195	66.9	0.01532	„	214	3.16	0.710
II	I	0.339	94.5	0.01280	8.473	162	2.99	0.631
July 6	II	0.235	92.2	0.01173	8.302	263	3.30	0.707
	III	0.176	90.8	0.01149	7.966	386	3.58	0.863
	IV	0.127	89.1	0.00822	7.656	661	4.06	1.024
III	I	0.172	88.6	0.01041	8.037	406	3.63	0.833
July 18	II	0.218	93.6	0.01095	„	298	3.39	0.758
	III	0.206	96.5	0.01035	„	337	3.48	0.783
	IV	0.172	94.7	0.00982	„	439	3.69	0.859
	V	0.0926	64.8	0.00768	7.786	659	4.05	0.982

Discussion of Results.

It is seen that the deviation of the points at the lower temperatures is greater than at the higher ones. This probably comes from the greater difficulty in handling the calorimeter and in measuring at the lower pressures.

In a graphical representation our observations give points, which tend to lie on a straight line, which passes through the origin; between 2°.5 and 4°.0 K. one has about $C_{sat} = 0.222\ T$. Within our range of temperature we may with reference to the accuracy attained put $C_{sat} = C_p$.

There would be no sense in applying this relation between larger limits. At first the value of C_{sat} must rise much more rapidly with the temperature above 4° K., because it becomes infinite at $T = 5°.19$ K. (critical temperature); and moreover it is improbable that C_p will approach to zero in proportion to T.

If one considers the deficiency in percentages of the specific heat C_v of a monatomic liquefied gas according to the law of DULONG and

PETIT as a measure of the degeneration of the liquid and if 2 is taken as the atomic weight of liquid hydrogen, then, neglecting the difference between C_p and C_v, an analogous degeneration seems to take place with liquid helium and liquid hydrogen; further it appears that as a function of the reduced temperature the degeneration for helium is greater than that for hydrogen. Examining this difference more closely, it must be kept in view, that $C_p - C_v$ is the same for different substances at the same reduced state, if the law of corresponding states holds, but that this difference by *deviations* [1]) will have different values at the same reduced state for helium and hydrogen; as $C_p - C_v$ at the examined reduced states is rather large, these deviations must be taken into account rather strongly.

[1]) Leiden Comm. Suppl. N°. 23, § 38.

Physics. — *"On crystallized Nitrogen"*. By D. VORLÄNDER and W. H. KEESOM. (Communication N°. 182*c* from the Physical Laboratory at Leiden).

(Communicated at the meeting of June 26, 1926).

§ 1. *Introduction.* According to WAHL [1]) nitrogen when solidifying out of the liquid state at —210° crystallizes in the cubic system and is optically completely isotropic. On the other hand DE SMEDT and KEESOM [2]) have examined the structure of crystallized nitrogen by Röntgen analysis at —253° and found that in all probability at this lower temperature the nitrogen is not cubic. EUCKEN [3]), as well as KEESOM and KAMERLINGH ONNES [4]) had already established the fact that at 35°.5 K. a transformation takes place.

According to these results nitrogen would belong to that important class of dimorphic substances, in which the cubic form which arises at higher temperature passes at falling temperature enantiotropically into an optically anisotropic form. Nitrogen would therefore behave in the same way as white phosphorus [5]).

§ 2. *Micro-observations at temperatures down to* —253° C. We investigated whether by means of a microscope in parallel polarised light a characteristic transformation isotropic $\rightleftarrows$ anisotropic could be ascertained in nitrogen and in some other substances crystallizing in the cubic system when cooled to —253°. After overcoming the initial difficulties we were able to observe the substances, cooled in a stream of cold hydrogen or plunged in liquid hydrogen, between crossed nicols. In the mean time experiments with cubic substances [6]) of very various nature, and further observations on a few anisotropic substances had shown that at these low temperatures a large harvest of new forms was not to be anticipated; the transformation isotropic → anisotropic in particular, did not take place. Ammoniumchloride for which SIMON [7]) thought to have found a trans-

1) W. WAHL, Proc. Roy. Soc. A **87**, 371, 1912; A **88**, 61, 1913; Z.S. physik. Chem. **84**, 106, 1913.

2) Physica **5**, 344, 1925; Leiden Comm. N°. 178*b*.

3) A. EUCKEN, Verh. d. D. physik. Ges. **18**, 4, 1916.

4) These Proc. **18**, 1247, 1916; Leiden Comm. N°. 149*a*, p. 8, note 2.

5) D. VORLÄNDER, Ber. d. D. chem. Ges. **58**, 1802, 1925.

6) Gold, iron, silver, diamond, KCl, KBr, KJ, NaCl, NH_4Cl, TlCl, TlBr, CaF_2, K_2PtCl_6, As_2O_3, $(NH_4)_2\,SnCl_6$, $(NH_4)_2\,TeCl_6$, ferric-ammonium-, thallium- and potassium-alum, spinel, corundum, ruby, zincblende, analcym, pyrope.

7) F. SIMON, Ann. d. Phys. (4) **65**, 264, 1922; c.f. W. EITEL, Fortschr. d. Mineralogie **8**, **34**, 1923, according to whom the transformation of ammonium chloride "extends over a large temperature range and cannot be demonstrated dilatometrically".

formation point at —243° while measuring the molecular heat, remains down to —253° optically isotropic; common and blue rock salt remain unchanged; pleochroïc solid crystals retain their pleochroïsm [1]); ice remains hexagonal between 0° and —253°.

What we observed in the anisotropic substances [2]) through the microscope was principally some changes which gradually took place in the volume and the double refraction with rising and falling temperatures and the formation of cracks in the material, but no ordinary dimorphic transformations, which should take place with a complete change in the crystalline structure with a distinct, even if slow, growth, from the one to the other form. In this way we observed under the microscope for the space of $^1/_2$ to $^3/_4$ hour several preparations plunged in liquid hydrogen. This does not entirely exclude the possibility of transformations still taking place, but we regard it as highly improbable, when we consider further that during a very slow re-heating of the substances no transformations were observable.

§ 3. *The preparation of pure nitrogen.* For these experiments with nitrogen the vital thing was to procure the gas absolutely pure. This was done from ammonium nitrite, by a gradual introduction of a concentrated aqueous solution of sodium nitrite (1 equivalent weight) through a dropping funnel into an aqueous solution of ammonium sulphate (2.5 equivalent weight) and yellow potassium chromate [3]) (1 equivalent weight) heated on a water bath to 80°—95° [4]). To expel the air from the apparatus (the solutions having been previously boiled) two of these nitrogen generators were connected in series; they were set working one after the other, after having been washed through with commercial nitrogen and the whole of the rest of the apparatus evacuated. This consisted of glass tubes sealed together, with the exception of the taps to the oxidation tube, which were closed by mercury. The nitrite nitrogen from the generators ran through the following apparatus:

1. two vessels cooled in icewater, containing 5 volumes saturated solution of potassium bichromate and 1 volume concentrated sulfuric acid, to oxidise oxides of nitrogen;
2. two tubes filled with pieces of sodium- or potassium hydroxide;
3. two drying vessels cooled with liquid air, to freeze out water vapour, NO, etc.

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[1]) Undercooled pleochroïtic mono-axial layers of liquid crystals (cyano benzalaminocinnamic active amyl ester) changed into a fine net-work with very irregular boundaries, while loosing their pleochroïsm.

[2]) Sulphur, phosphorus, bromine, iodine, $Pb(NO_3)_2$, CdJ_2, SbJ_3, TlJ, Hg_2Cl_2, K_2PtCl_4, $AsBr_3$, H_3BO_3, ZnO, gypsum, arragonite, Iceland spar, quartz, beryl, tourmaline, dichroïte, JCN, $(CH_3)_4NJ$, $Hg(CN)_2$, CCl_4, $CHBr_3$, normal heneicosane, hexadecylene, norm. valeric acid, stearic acid, elaidic acid, tristearine, norm. hexylalcohol, cyclohexanone, methyloxalate.

[3]) C.f. VON KNORRE, Chem. Industrie **25**, 531 and 550, 1902.

[4]) C.f. H. KAMERLINGH ONNES and A. TH. VAN URK. Leiden Comm. N°. 169*d* and A. TH. VAN URK, Diss. Leiden, 1924, p. 21.

4. an oxidation tube heated in a gas oven to low red heat, provided for $^4/_5$ of its length with copper gauze and for $^1/_5$ with CuO;

5. two drying vessels cooled outside with liquid air (one of them was a spiral glass tube);

6. two condensation vessels cooled outside by liquid air, boiling under reduced pressure; the nitrogen was liquefied in the first of these vessels and then distilled over into the second vessel (after the surrounding liquid air had been brought back to atmospheric pressure).

The nitrogen was then distilled over, into a previously evacuated glass bulb, for storing, or directly into a 25 to 30 cm^3 pipette *p* (fig. 1) with a tap connected to the apparatus, to which was sealed an observation chamber *c* made of plane parallel glass plates of about 1 cm diameter [1]). For the final filling the pipette and chamber were highly evacuated (6 times) and filled with nitrogen. The nitrogen condensed in the first condensation vessel (No. 6) formed a clear, colourless liquid which showed no trace of turbidity and when distilled off left no trace of foreign matter. After washing, the nitrogen was fractionally distilled over from the first condensation vessel into the second (No. 6); then the pipette with observation chamber was filled with it, up to about 40 cm pressure. This nitrogen contained no oxygen or oxide of nitrogen (tested with alkaline solution of pyrogallol and when admitting air, with acidified potassium iodide). By sealing on in front of the pipette a glass bulb (*b*) filled with nitrogen under increased pressure (80 cm) the possibility of oxygen from the outer atmosphere penetrating through the tap of the pipette was prevented. In the observations with the polarisation microscope the chamber was placed in a cryostat glass, of which the inner wall was cylindrical, while on the outer wall at the height of the chamber two plane parallel glass windows were made. The nitrogen crystallized on being cooled with hydrogen, to a layer of 0.2 to 0.3 mm thickness in the observation chamber [2]).

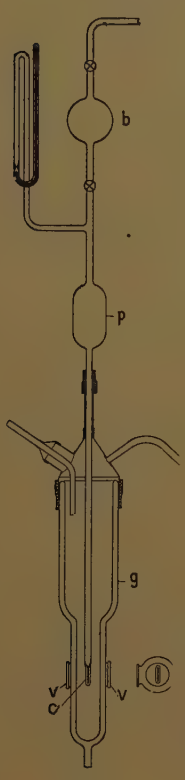

Fig. 1.

§ 4. *Observations of the crystallized nitrogen in the polarisation microscope.* As the nitrogen cristallizes out from the liquid by cooling, the first particles of solid nitrogen still moving in the liquid appear to be double refracting. Shortly after the whole mass has become doubly refracting and solid, a marked contraction, of roughly from $^3/_4$ to $^4/_5$ of the original

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[1]) Simple mercury manometers, acting at the same time as safety valves, were placed: *a.* between the generator and N°. 1; *b.* between N°. 5 and N°. 6; *c.* between the first and the second vessel of N°. 6; *d.* behind N°. 6; *e.* at the pipette.

[2]) We wish to express our thanks to Mr. W. PLOEG for his help in preparing the pure nitrogen.

volume of the fluid takes place. The solid optically anisotropic mass is strongly deformed, while forming tubular and vermiform holes, from the surface of which often innumerable very minute and branched cracks and holes spread out, which between crossed nicols appear black. Pleochroïsm could not be observed at the transparent surfaces when rotating the polarisor, the other nicol being removed. The mass which seems to be principally fibrous and in some places tabular, shows no perceivable definite structure, so that the existence of definite angles or extinction directions could not be observed. On further cooling the double refraction changed. At the introduction of a gypsum plate 1st order, red polarisation colours changed into yellow, violet into blue. This change takes place gradually; at the same time a few fairly rapid changes were seen, in which a fine seam appeared to run over the mass. A distinct transformation from one optically anisotropic form into another could not be ascertained. A transformation from an isotropic to an anisotropic phase from the first cristallization at —210° to —253°, is excluded.

With rising temperature first the changes of the polarisation colours in the opposite sense are seen; after that, on approaching the melting point, the holes partly fill up, so that a clear tabular crystalline mass forms, in which no special direction of growth can be ascertained, as is the case with falling temperature, as apparently the crystalline particles are orientated and grown together in every possible direction. The pattern that is formed is very peculiar, we do not remember ever to have seen one like it in ordinary crystalline masses of other substances. Now the melting begins very speedily, during which the solid "tables" remain optically anisotropic to the last particle, floating in liquid nitrogen. It thus appears that here too there is no isotropic phase. These observations make it quite clear that the double refraction which was observed could not be only due to deformation, caused by the contraction of an isotropic form.

§ 5. By the observations described in § 4 the cooling in the stream of cold hydrogen and in liquid hydrogen respectively took place fairly quickly. As Dr. TERPSTRA pointed out to us it was possible that with rapid cooling anomalous forms would arise. We have therefore repeated the observations to about — 218° in a bath of oxygen, boiling under reduced pressure. By regulating the pressure of this we could regulate the rapidity of the crystallization as well as that of the re-heating.

In these experiments, in which Dr. TERPSTRA was so kind as to take part, the nitrogen was observed: 1. with an ordinary microscope, 2. with crossed nicols in // light, 3. with crossed nicols and a gypsum plate red 1st order in // light, 4. in a conoscope with convergent light.

The following observations were made:

1. With the microscope the coagulation in the field of vision could

be seen gradually proceeding, mostly in a few // fields with progressing boundaries about ⊥ to the meniscus of the liquid. The fields were often directly distinguishable on account of the difference in intensity of the light let through.

2. Between crossed nicols the differences in intensity of the fields could not be fixed with certainty. Neither could differences during synchronic rotation of the nicols be observed. The thickness of the crystal layer was about 0.3 mm, the visible field formed a circle of 5 mm diameter.

3. With the gypsum plate changes of colour usually occurred, which indicated a very weak double refraction. Occasionally a field was observed without perceptible double refraction, between others which were distinctly doubly refracting.

4. Conoscopic investigation yielded only negative results. Neither cross, rings or hyperboles were observable. Rotation of the analysor showed that optical activity played no part.

A repetition of observations 1—3 by Messrs ESBACH and PLOEG with nitrogen prepared from sodium azide led to the same conclusions.

§ 6. *Conclusion.* The observations given in §§ 4 and 5 lead to the conclusion that nitrogen immediately at the melting point is doubly refracting. The double refraction is extremely small.

The fact that WAHL did not find any double refraction may be due to his having used a layer only $^1/_{20}$ mm in thickness, and making his observations only with crossed nicols, and not with the more sensitive method of the gypsum plate.

Crystallized nitrogen cannot therefore be compared to the dimorphic white phosphorus. Its anisotropic structure corresponds better with the multivarious chemical properties of nitrogen than would be the case with the regular structure so far assumed.

§ 7. *Observations on crystallized argon.* In agreement with WAHL's results [1]) and with the Röntgen analysis of F. SIMON and CLARA VON SIMSON [2]) and of DE SMEDT and KEESOM [3]) argon according to our experiments is regular to —253°. The manner of crystallizing is very different from that of nitrogen. Argon freezes to an optically isotropic homogeneous mass, which with more intense cooling shrinks as a whole and often springs off from the walls of the observation chamber without becoming doubly refracting.

On being gradually heated the crystallized argon of our experiments began to sublimate strongly. The glass wall visible above the crystalline

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[1]) See p. 1069, note 1.

[2]) F. SIMON and CLARA VON SIMSON, Z.s. f. Physik **25**, 160, 1924.

[3]) See p. 1069, note 2.

mass was covered with small isotropic crystals, as well as with needles which were grown together at right angles to each other. After that the argon melted.

To conclude we are glad to express our thanks to Mr. V. ESBACH for the great help that he has given us, and to H. VORLÄNDER for his help in part of these experiments.

Physiology. — *"The method for graphic determination of the total gaseous exchange in man during muscular activity."* By J. G. DUSSER DE BARENNE and G. C. E. BURGER. (Communicated by Prof. H. ZWAARDEMAKER.)

(Communicated at the meeting of May 29, 1926)

In previous communications [1], [2], [3], [4], [5] we have decribed a method for the graphic determination of the total gaseous exchange in man and animal at rest, i.e. the consumption of O_2, the output of CO_2, the respiratory quotient and ventilation. The advantages of this method are the following: 1. gas analyses are not necessary; 2. important saving in time; 3. the experimental data are and remain available in the form of curves for control and reproduction. On comparing this method with different gas analytical methods, e.g. with the most accurate one known so far, that of ZUNTZ—GEPPERT, it appeared to give very well corresponding results; our method is therefore reliable and accurate.

On having determined this, we tried to use this method also for the determination of the gaseous exchange of man during muscular exercise. We believe to have succeeded; with the following method it is possible to determine the gaseous exchange of a subject graphically before, during and after muscular activity.

The principle of this extension of our method is that through the respiration of the subject with the aid of our ,,trap apparatus" and its volume recorder and a second volume recorder filled with soda-lime, ,,CO_2-curves" and ,,O_2-curves" are written alternately below each other on a kymograph. From the descent of the O_2-curves and the descent, resp. rise of the CO_2-curves within a definite period of time, one can obtain the data enabling to determine with a very simple calculation the desired figures on the gaseous exchange during the separate experimental periods.

This method will be best described by referring to the subjoined diagram. (Fig. 1).

M is the mouthpiece which the subject takes in the mouth, causing him to breath (after blocking the nose with a nose clip) through the valves *I* and *E*. These communicate through wide rubber tubes *a* and *b* with the big three-way taps *II* and *III*. In the position

1) Klinische Wochenschrift 1924, N°. 10.
2) Journal of Physiology **59**, 1924, p. 17.
3) These Proceedings **27**, p. 382.
4) Klinische Wochenschrift 1925, N°. 2.
5) Zeitschrift f. d. gesammte experiment. Medizin, **59**, 1926, p. 130.

as given in figure 1, the subject inspires along valve *I*, tube *a*, tap *II* and pipe *e* from volume recorder 1 (pattern of KROGH) and expires along valve *E*, tube *b*, tap *III* and pipe *c* into spirometer *A* of our apparatus, the latter having been indicated diagrammatically by the concentric circles *A* and *B*. With every quantity of expired air, going along the way mentioned above, into spirometer *A* and causing it to rise, an equal volume of air

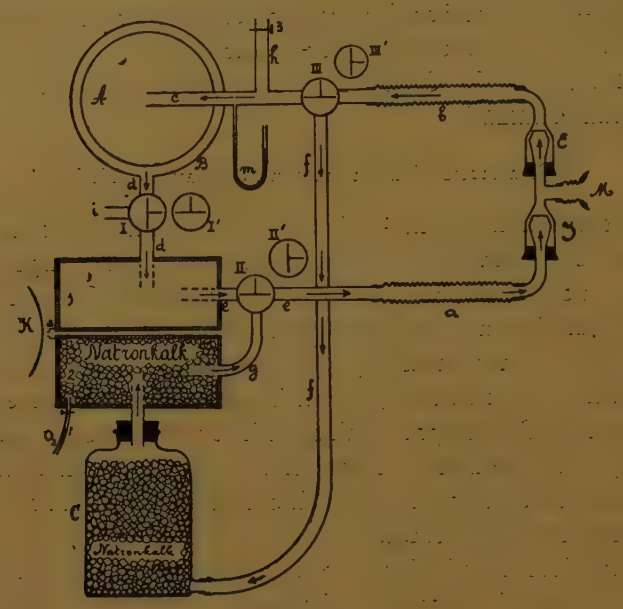


Fig. 1.

from cylinder *B*, (closing off spirometer *A* hermetically) is expelled into volume recorder 1 along pipe *d* and threeway tap *I*. For every one conversant with the working of our respiratory apparatus it will be clear that when taps *II* and *III* are in the position as shown on the fig. 1 the respiration of the subject writes a CO_2-curve on kymograph *K*. If the three-way taps *II* and *III* are turned 90° to the right, bringing them into position *II'* and *III'*, the trapapparatus is closed and the subject inspires from volumerecorder 2 along pipe *g*, tap *II*, tube *a* and valve *I* and expires into this volumerecorder along valve *E*, tube *b*, tap *III*, pipe *f* and flask *C*. As the tray of volumerecorder *II* and flask *C* are filled with soda-lime, the CO_2 from the expired air will be absorbed if the subject communicates with volumerecorder 2 and through the respiration an ordinary O_2-curve will be written by its writing point. During this O_2-period the trapapparatus is closed and the writing point of volume-recorder 1 makes a control line. Reversibly, if the subject communicates with the trapapparatus and volumerecorder 2 a CO_2-curve is written on the kymograph through his respiration; during this period the pointer of volumerecorder 2 writes a control line.

From the above it will be clear that through turning simultaneously the three- way taps *II* and *III*, then to the left and then to the right, the subject can be made to "write" alternatively then a CO_2-curve and then an O_2-curve. In this way we get two curves, which schematically look like figure 2.

From the descent of these O_2-curves and the descent resp. rise of the CO_2-curves within a definite period of time, we get all data to calculate

the desired figures on the gaseous exchange during these periods. It is clear that it is thus also possible to determine the gaseous exchange of man at rest, but we wished to use this method in the first place for experiments on work.

The capacity of spirometer *A* of our apparatus for the human subject is about 130 litres, which is amply sufficient during rest to allow for periods

Fig. 2.

of 15—20 minutes. But during work, when the pulmonary ventilation is often 50—80 litres a minute, spirometer *A* is filled in 2—3 minutes and with maximal muscular exercise, still quicker. To render this method useful for experiments on work, spirometer *A* must be emptied quickly when filled with expired air during the registration of an O_2-curve, so as to be empty before the next CO_2-curve begins.

This condition we have been able to satisfy in a simple way. As soon as the moment has arrived on which the spirometer must be emptied, 3-way tap *I* is turned 90° to the left during the tracing of an O_2-curve (position *I'*), causing volume recorder *I* to be closed off and cylinder *B* to communicate with the outside air along the central piece of tube *d*, top *I* and tube *i*. Now through tube *h*, a side tube of *c*, which otherwise is closed at 3 through a tap or stop, we empty spirometer *A* by suction. Spirometer *A* is thus emptied which is possible, because during its descent a corresponding quantity of air is sucked into cylinder *B* along tube *i*, tap *I* and tube *d*.

We use a Phön apparatus for this suction. One of its two perforated sides is hermetically closed, the other one has a tight fitting funnel, on which fits the cork at the end of tube *h*. When the Phön apparatus is put into action through turning the button, it wil obtain its air, not from the outside, but from spirometer *A*. The Phön apparatus used by us empties our spirometer in about one minute.

Where we generally take records of periods of from 1.5 to 3 minutes, it is thus possible, during the writing of an O_2-curve, to empty spirometer *A* (filled with expired air) so that it becomes available for the next CO_2-curve.

We fill the volume-recorder 2, with which the O_2 curves are written, with O_2 from an O_2-cylinder, causing enough O_2 to be available for O_2-periods of 2—3 minutes, even during heavy exercise. After an O_2-period, and during the writing of a CO_2-curve, volume recorder 2 may be filled along tube O_2 making enough O_2 available again for the next O_2-curves.

The results and curves obtained from similar experiments on work, appear from figure 3, which shows two sets of curves in a complete experiment on the gaseous exchange, obtained by the above method (see fig 3).

In this preliminary communication we will have to be satisfied with the discussion of a single item. During the registration of the O_2-curves, the subject inspires O_2, during the CO_2-curves, ordinary atmospheric air. One might suppose that the alternate breathing of gas mixtures of different composition, could lead to inaccurateness. For the most obvious thing to think is that during inspiration of the nearly pure O_2 from volume-recorder 2, a certain extra quantity of O_2 will be dissolved physically in the body and a certain quantity of N_2 will be expelled, in accordance with the smaller partial tension of the N_2 in the O_2 mixture in relation to air; on inspiring ordinary air during registration of a CO_2-curve, one would expect the opposite. Where the absorptioncoefficients of O_2 and N_2 in blood and water are approximately as 1 to 2, one might easily suppose that through this factor the figures for the quantities of O_2 used during the respective periods of the experiment, are not accurately corresponding with the quantities used by the organism.

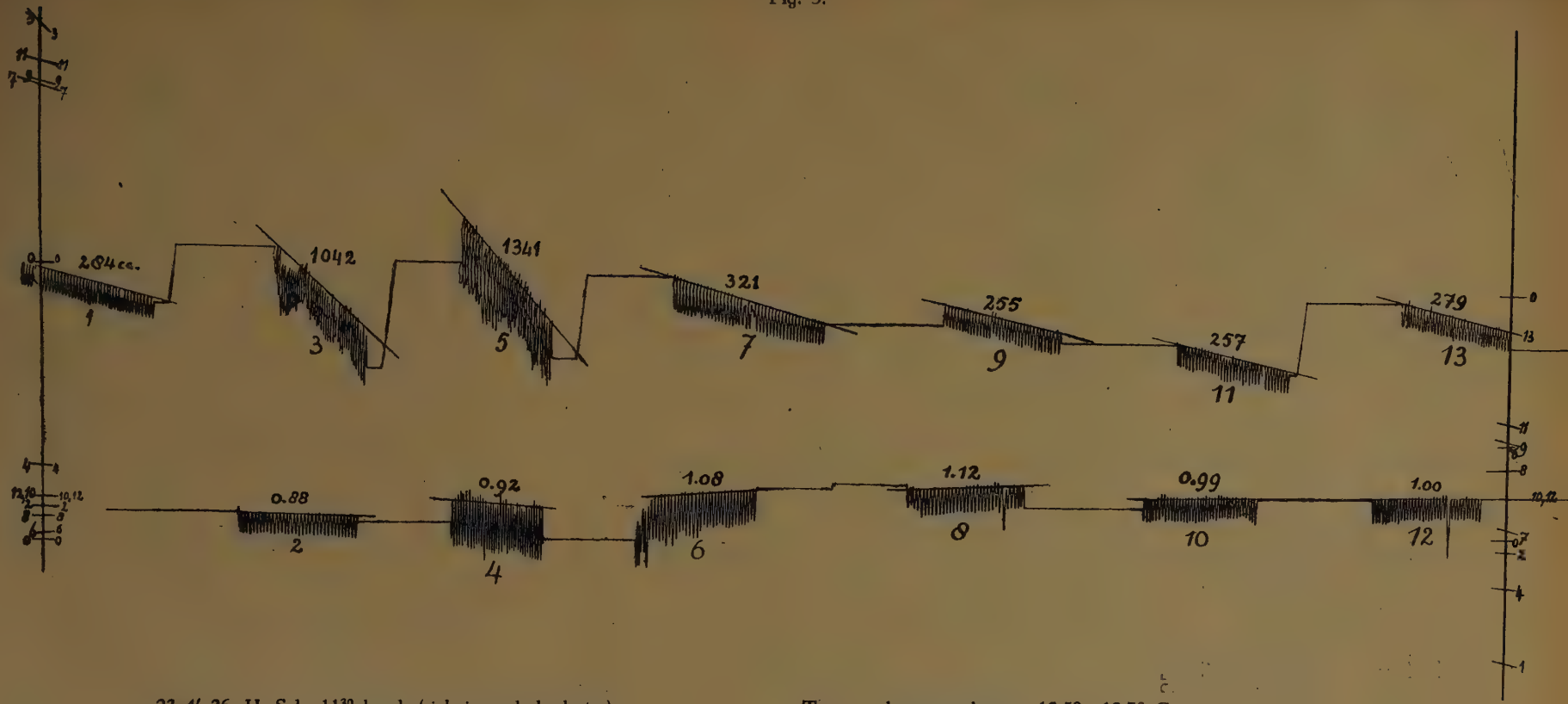
To examine this point we have determined several times graphically in different subjects at rest the O_2-intake during periods of two minutes with alternate breathing of O_2 and ordinary air, now after a long foreperiod on ordinary air, then on O_2. The result of this control has been that in our method in inspiring alternately O_2 and air during periods of 1.5—3 minutes, this factor, if present, does not interfere. This is shown by the following table.

The average of these two groups of control experiments is the same. Moreover, the differences which have been found, and which are very small for experiments on gaseous exchange, are in several cases in opposite direction; a systematic deviation therefore not observed. The beautiful

Consumption of O_2 in cc. per minute (reduced to 0° and 760 mm. Hg.)

After adaptation to O_2	After adaptation to air
280.1	274.6
279.2	279.2
297.7	290.1
295.6	296.8
299.6	309.7
Av. 290.5	Av. 290.1

Fig. 3.



23.4'.26. H. Sch. 11^{30} lunch (rich in carbohydrates).
Fore period: sitting quietly 35 min. in easy chair.
6 min. adaptation on O_2.
curves 1—13: 3^{37}—4^{02} (25 min.).
exercise during curves 3, 4 en 5 ($4^1/_2$ min. standing running 164 steps per min.)

Temp. volumerecorders = 12,5°—12,7° C.
Corrected bar. = 758 mm. Hg.
Factor = 0,939.
1000 cc. of each volumerecorder = 17 mm. on the original curves.

Enlargement of this fig. in comparison with the original curves is 0.43.
The R. Q. given above CO_2-curve 6 must be **1.17** instead of 1.08.

analogy between the two corresponding numbers in each experiment also shows the usefulness and accuracy of the method.

Our method is so simple in its execution, that one can do the whole experiment with two people, the subject and the experimentalist. Further, it entails a very valuable saving in time. The calculation of the curves given in fig. 3, has not taken 35 minutes. Therefore, in about 35 minutes, the different figures of the O_2-consumption and the 6 given respiratory quotients were known. If we estimate the time necessary for the determination of these data in using one of the gas analytic methods, there would have been necessary for the six gas analyses (taking it of course they were done without interval one after another by one analyst) 6×20 minutes or 2 hours. Adding to this the time necessary for the measurement of the ventilations during the 6 periods and the calculations of the respiratory quotients from these gas analyses and ventilations, these calculations taking much more time than in our method, one would have to reckon $2\frac{3}{4}$ hours at least for obtaining the same data by gas analysis.

Briefly, we may thus summarize this preliminary communication: The above described method is an extension of the method given in previous communications for the graphic determination of the total gaseous exchange. It permits to determine this in man also in the so-called excercise-experiment, i.e. before, during and after muscular activity. The experiment may be continued as long as the experimentalist desires. The method is accurate and offers several advantages against the gas analytic methods hitherto used in this kind of research.

(Physiology Laboratory of the State University, Utrecht.)

(May 1926.)

Histology. — *"An Inquiry into the Physico-chemical Structure of the Collagenic Substance. I. On the Spiral Arrangement and on the so-called Hygroscopic Torsion of Collagenic Bundles of Tendons."* By Prof. G. C. HERINGA and Miss H. A. LOHR. (Communicated by Prof. J. BOEKE.)

(Communicated at the meeting of June 26, 1926).

Being struck by the peculiar sinuous course of the collagenic fibres that we found again and again in all our preparations (as well in the broad tape-shaped bundles as in the most delicate, in dark field optically homogeneous filaments) we have made a close inspection of the arrangement of the bundles in the tendon. Our endeavour resulted in a complete confirmation of WOERDEMAN's [1]) description. That collagenic bundles are intertwined in the tendon in groups, became evident from the following observations:

1°. In longitudinal sections of a tendon with purely parallel fibres (mouse-tail) the "azimuth blende" [2]) showed us in dark field the interlaced course on a different level of the fibres.

2°. In transverse sections (mouse-tail tendon, achilles tendon human embryo) it is seen on turning the screw of the micrometer that bundles are coiling in groups round each other, as described by WOERDEMAN. It is remarkable that the last-named object displays the peculiarity that all over the transverse section complexes occur side by side twisting to the right and to the left.

3°. The spiral interlacement of the bundles is demonstrated by the following experiment even more conclusively than by the two preceding facts:

A tendon with parallel fibres (extensor of the toe of a cow) is firmly clamped by a screw. The other end is held by the hand or simply hangs loose. A slender, pointed lancet is inserted superficially into the tendon with the blunt edge of the blade towards the experimenter, and the flat side parallel to the long axis of the tendon. By pushing the knife down so far that the point reappears on the other side a slender bundle of fibres is split off from the tendon. Now the knife is drawn back, so that the split bundle is detached from the tissue. It then appears that under this operation the free extremity of the tendon twirls round as a rope made of twisted fibres does when used for a similar experiment.

A repetition of the experiment, not with the entire tendon, but with a detached bundle showed that the phenomenon became still more apparent according as the

[1]) H. W. WOERDEMAN, Histologisch onderzoek naar den fibrillairen bouw van eenige cellen en weefsels. Diss. A'dam, 1921.

[2]) Cf. G. C. HERINGA and H. A. LOHR. These Proceedings, Vol. 28, p. 509.

bundle is thinner. From this we conclude that intertwining of the bundles, down to increasingly finer texture, is repeated over and over again, which in fact tallies completely with WOERDEMAN's observation, as well as with ours of the microscopic preparation (transverse section).

There is no doubt, then, but that in the tendon there exists a spiral intertwinement of the bundles. A similar condition of the bundles is found in microscopic preparations of the skin and of the loose connective tissue (umbilical cord, mesenterium etc.), in which the sinuous structure of the bundles can be watched in one field of vision as the spindle is so much smaller.

Seemingly conflicting with all these observations was the fact that in a great number of largely magnified photos of mouse-tail tendons swollen in acids a very distinct parallel striation was noticeable. While occupying ourselves with the question in how far a detorsion under the influence of the acid-swelling might come into play here, HATSCHEK's publication in the Kolloid Zeitschrift 1925 was brought under our notice by Dr. BUNGENBERG DE JONG.

HATSCHEK has bent rectangularly moulded gelatin rods and held them in fixation so long that deformation became permanent. Through desiccation the curvature was then rendered firmer. Swelling caused its detorsion. HATSCHEK also twisted similar rods into spirals; desiccation induced further torsion, swelling brought detorsion. Control-experiments with nondeformed gelatin-models of a definite mould exhibited no morphological change, but a homomorphic change of volume excepting phenomena of "case-hardening".

To put it shortly: HATSCHEK's experiments go to show that desiccation gives permanence to a previously effected deformation of the gelatin, and that swelling causes it to disappear.

We have repeated these experiments with quite similar results in so far as a simple curvature is concerned. Only we feel urged to add two observations to HATSCHEK's description.

1°. that the morphological change is introduced by a shortening on the concave side without an increase, nay rather with a decrease of the concavity. At the same time the short terminal surfaces are converging towards the concave side. This clearly proves, what in strictness could be anticipated, viz. that the warping of the rod rests on traction emanating from the side compressed during the deformation process. This lends support to the conception that the phenomenon originates entirely from the loss of water through the micellae, which loss asserts itself most where the micellae are most accumulated. This conception also makes the second fact intelligible, which we will mention as an addition to HATSCHEK's description, viz. that curvature caused by desiccation begins only after a considerable, fairly homomorphic diminution: the influence of the disposition of the micellae is more distinctly visible according as the latter are packed

more closely (had we rather say: are more closely accumulated?) through evaporation of intermicellae water.

As we endeavoured to find a parallel between the twisted gelatin and our tendons, we have started experiments on the behaviour of tendons during swelling and desiccation.

Again we took a piece of a cow's tendon (toe-extensor), 10 c.m. in length. Judging from its outward appearance its fibres ran parallel. At the top it was clamped, at the lower end it was provided with a pointer and dipped vertically in a weak acetic-acid solution. The first experiment yielded directly positive results. After two or three hours the pointer indicated a convolution of more than 90°. This experiment was repeated some times, modified in such a sense that lateral curvatures, which greatly vitiated the readings, were counteracted by a balance at the lower extremity. The same torsion phenomenon was noted every time. But the deflections were seldom higher than 15°—30°, while, moreover an insight into the nature of the problem was impeded by the fact that swelling caused pieces of the same tendon to wind alternately to the right or to the left. This fact invalidated our estimation of the direction of the torsion. Besides this there was, however, the fact that convolution also took place with desiccation (the tendon, mounted in the same way being suspended in the air).

The idea that in such a complex as an entire tendon the combination of elements with a tendency to opposite torsions might play an influence upon the result of the experiment, prompted us on Dr. KOLKMEYER's suggestion to repeat the experiment, with small fibre-bundles, isolated from a tendon in the way above described. Now all at once convolutions made their appearance, not seldom with deflections of 360° or 2 × 360 degrees. That nevertheless no regularity was found in the size of the angle of torsion, so that a quantitative analysis of the phenomenon is not possible for the present, is quite conceivable, if we reflect that our splitting-method is still a rough-and-ready method, and how much the result is affected, as may be seen during the experiment, by adhering rough particles and probably also by mechanical lesions to the bundles. With these delicate filaments in the same object the same torsions to the right and to the left manifested themselves either side by side or sometimes in succession. It was evident, however, that the torsion caused by swelling leads in the first instance to a winding down of a torsion that always appears at desiccation. This torsion is the fundamentally equally important counterpart of the detorsion caused by swelling. It is very conspicuous at the tapeshaped and slightly crinkled fibre-bundles.

Thus we establish the fact that the collagenic fibres present with desiccation a torsion opposite to that presented with swelling; they behave like the gelatin in HATSCHEK's experiments. Besides the observations described by HATSCHEK we know already for a long time a similar phenomenon from quite another quarter. This may the more readily be correlated with our torsion of collagen, as both relate in the same degree to fibrous substances furnished by nature. We now think of the phenomenon designated by botanists as "hygroscopic torsion".

It had already long been known that many plant-parts flex or twist on (physiological) desiccation, when DARWIN [1]) pointed out that the origin of the movements

[1]) F. DARWIN, Transactions Lin. Soc., 2nd Series I, 149, 1876.

was to be looked for in the properties of the cells. ZIMMERMANN [1]), coinciding with NÄGELI's hypothesis, established the relation between the disposition of the micellae and the location of the „shrivelling-ellipsoid", and lastly STEINBRINCK [2]) found an explanation by means of extremely ingenious models for the various morphological changes (flexure, spiralization, torsion) originaling from the disposition of the micellae. Without entering into details we wish to remind the reader that STEINBRINCK has demonstrated that, when different micellar systems are united into one system in such a way that their several shrivelling-ellipsoids do not coincide, the deformation agrees on all points with the resultant of the tensions appearing in situ.

When applying STEINBRINCK's reasoning to HATSCHEK's flexed gelatin-rod, it will be seen directly that desiccation, indeed, induces reinforcement of the curvature, anyhow if we assume provisionally, that the gelatin-micellae, like those of plant-fibres, are elongated anisotropic rods, and that these micellae, without changing their position as a whole, are placed by the deformation in such a way that their long sides are turned outward on the convex side, and on the concave side one of their short ones, assuming also that, again as in the case of the plant-fibre, the long axis of the shrivelling ellipsoid coincides with that of the micellae.

For when we examine a slice defined by two parallel planes, taken from the middle of the gelatin-rod *a b c d*, in which there are on the convex side *a b* two micellae with their long axis parallel with the axis of the rod, and on the concave side *c d* two micellae normal to it, then it will follow directly from the position of the shrivelling-ellipsoids that *c d* is shortened more than *a b*, which when totalized must bring on a flexion of the whole rod.

A similar reasoning may afford an explanation of the desiccation-torsion, viz. when we imagine the twisted object, from which we started, divided by parallel planes into a number of thin slices. STEINBRINCK's scheme, which is in every way admissible for botanical objects, tested by polarization-microscopic investigations, is, therefore, also adapted to HATSCHEK's gelatin-model if the gelatin-micellae are anisotropic. Not only in this case, however, When suggesting the above mode of explanation we started from the ascertained fact, (without taking into account the anisotropism of the separate micellae), that on the compressed side the micellae are packed more closely. This, indeed, could readily be verified by mixing the liquid gelatin before the moulding with gold-sol, and by counting the particles by means of an Ehrlich-blende. Then it should only be remembered that for the explanation it is immaterial whether a single particle is answerable for a definite shrivelling or whether it is the sum of the mean decrease of many particles, in order to see that the two explanations coincide completely. From either starting-point we arrive at the conclusion that with dehydration the micellar-anisotropic structure of any object brings about a change in

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[1]) A. ZIMMERMANN, Mechan. Einrichtungen zur Verbreitung der Sauren und Früchte, etc. — Jahrb. f. Wiss. Botanik 12, 1881.

[2]) C. STEINBRINCK, Ueber Schrumpfungen und Kohäsionsmechanisme der Pflanzen. — Biol. C. Blatt, 26, 1906.

the outward form, agreeing with an accentuation of the micellar arrangement.

Lastly we have still to observe that this formula is reversible in the sense that with evenly arranged micellae loss of water can bring about deformation only when the micellae are anisotropic.

After this lengthy expatiation let us revert to our own object. It is difficult now to interpret the swelling- and the desiccation-torsion of the tendon, respectively the tendon-fibrils, otherwise than by assuming either that the micellae are arranged as a spiral or every micella possesses itself a spiralshaped anisotropism, or finally both possibilities together.

Thanks to the kindness of our co-workers we are so fortunate to be able to verify this hypothesis by two investigations respectively by HERINGA and MINNAERT and by HERINGA and KOLKMEIJER. The firstnamed authors [1]) starting from an optic phenomenon observed in tendon-sections with transmitted light, came to the conclusion that the micellae of AMBRONN are arranged spirally. The great regularity also, with which definite convolutions revealed themselves in the preparation, was a strong argument for the conclusion that we had not to do here with an orientation of the fibrils incited from without (i. e. mechanical force in connection with development): the phenomenon in fact being evenly superposed on all histological arrangement. It was evidently bound to the properties of the collagenic substance itself.

Once so far we could not omit going farther into the real colloid-chemical-micellar formation of the collagen, as had already been done before us by HERTZOG and KATZ and others. Dr. KOLKMEIJER [2]) was so kind as to assist us in our investigation. To our great satisfaction we could derive conclusions from the röntgenogram that really favour our hypothesis, as it appeared 1^0. that presumably plate-shaped crystallites are arranged invariably at a definite angle with the axis of the fibre; 2^0. that, moreover, a periodicity exists in the long axis of the fibre. These facts, especially when combined, are quite compatible with the conception of a spiral structure. Indeed, allied facts can be pointed out in the crystallography of inorganic nature, as according to KOLKMEIJER, BIJVOET and KARSSEN'S [3]) inquiries a similar spiral structure also occurs in sodium-chlorate, and sodium-bromide.

This is a pregnant conclusion, as it leads to many consequences, which, especially when compared with the most recent observations by KATZ [4]) on the splitting capacity and the röntgenspectrum of elongated gelatin, will not fail to influence also our view of the gelatinizing process. This, however, will require closer inspection.

As to collagen itself, we wish to point out in connection with the foregoing that, if the probable structure of the elementary cell of the collagenic substance, rendered probable by this röntgen-treatment, should be confirmed,

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1) G. C. HERINGA and M. MINNAERT, These Proceedings, p. 1087.

2) G. C. HERINGA and M. H. KOLKMEIJER, These Proceedings, p. 1092.

3) These Proceedings, 23, 644, 1920.

4) l. c.

then, in accordance with STEINBRINCK's schema, the arrangement of the crystallites inside this cell (to be imagined as anisotropic rods or plates) must, when dried, naturally lead to torsion of this units. Although the spatical composition of these elements into a crystalline whole, has not yet been accomplished, it may be deemed probable that the composition of all these elemental torsions will induce torsion of the whole fibre, in the same way as DARWIN (l.c.) transfers the torsion of the separate sklerenchymatic cells to the torsion of the plant-part.

From this it would again follow that the torsion of the collagen is owing to the arrangement of the crystallites as well as to the typical lyophil-colloid character of the collagenic substance.

Taking all in all the result of this inquiry on the one hand confirms the relations between the collagenic substance and colloid-chemistry, especially the thread-sols [1]), on the other hand it confirms the relation we looked for, between collagen and the mesomorphic substances.

So the röntgendiagram corroborates our previously formulated conception [2]) of the origin of the collagen substance. Moreover, it explains a fact that had hitherto puzzled us. As the reader will remember we described how, in the first instance, the collagenic fibres presented themselves as thin filaments singly defined in dark field, how they subsequently grow into rather broad, doubly contoured, but still optically homogeneous permanent tapes, and lastly how after this secondary optic inhomogeneities appeared, that, while ever multiplying, gave the final, very closely striated, typical, fibre-bundlelike aspect to the collagenic mass. Well then, it is quite conceivable in addition to what we now know of the crystallography of collagenic substance:

1°. that like asbestos this must have numberless longitudinal planes of cleavage;

2°. that this cleavage (cf. splitting of dried wood) will appear spontaneously with the apparently physiological, gradual dehydration of the micellae, so that in connection with the accompanying torsion the planes of cleavage also assume a twisted course.

and 3°. that consequently, when micellae arranged round one crystallization-centrum are getting older, the whole collagen mass built up by them wil be split up in a number of fibres twisted round each other.

From the Laboratory for Embryology and Histology of the State-University at Utrecht.

Utrecht, 26 June '26.

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1) "Stäbchen-Sole" after SZEGVARI.

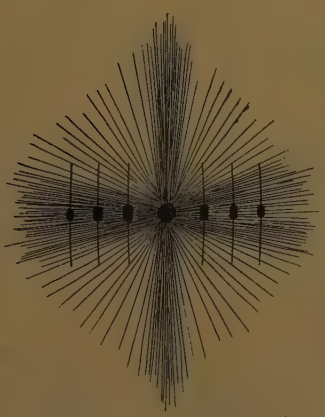
2) G. C. HERINGA and H. A. LOHR, These Proceedings.
G. C. HERINGA and H. A. LOHR, Bulletin d'Histologie, 1926.

Histology. — *"An Inquiry into the Physico-chemical Structure of the Collagenic Substance. II. On an Optic Phenomenon observed in sections of the Tendon".* By Prof. G. C. HERINGA and M. MINNAERT. (Communicated by Prof. J. BOEKE).

(Communicated at the meeting of June 26, 1926).

The structure described in this paper was found in longitudinal sections of a cow's tendon (toe-extensor) obtained by means of a microtome after previous freezing without embedding, and enclosed in a levulose-gelatine mixture. The preparations become completely transparent and look beautifully homogeneous.

We hold such a preparation close before our eyes, and look at a point-shaped lightsource. A "nitra-lamp" arranged at some metres' distance suits our purpose very well. Then we see a remarkable diffraction figure, which shows most distinctly when the room is darkened and the background is black. Let the direction of the fibres be horizontal. We see (Fig. I) round the light source a cross of light with four somewhat pencil-shaped arms. The vertical arms are rather narrow and long, the horizontal ones broader and shorter. In the horizontal lines of light, on either side of the lightsource there are to be seen three or four intensely bright, beautifully coloured diffraction-spectra. Moreover, the whole field is filled with a weak light radiating from the central light-point. The diffraction-spectra are disposed on a horizontal line, parallel to the longitudinal direction of the fibres.



direction of the fibres

Fig. 1.

In the best preparations the vertical arm is almost invisible. This part of the phenomenon must, therefore, be referred to fortuitous small fissures in the preparations in the direction of the fibres.

In order to determine the period of the structure, the light-source was replaced by a screen with a narrow slit, illuminated by sodium-light, and the angle was determined at which the diffraction-spectra could be seen. We found: 0.015 rad., which corresponds with a period of $\frac{0.589}{0.015}\mu = 40\,\mu$.

This value is, of course, an average. Probably periodic structures will appear in the preparation of differing λ. The optic examination gives

an average value over an area of the preparation equal to that of the pupil. The marked distinctness and the pure colour of the diffraction-spectra indicates, however, that the period can differ only little from one point to the other, or that portions with considerably deviating periods occur comparatively seldom.

This great regularity and the accurate orientation of the periodicity with respect to the longitudinal direction of the tendon renders it already improbable that some fortuitous ridging should have originated during the microtomy. Moreover, the direction of the cutting microtome was inclined at a large angle to the longitudinal direction of the fibres.

Now it was our desire to make out in a general sense whether the optic phenomenon was due to a ridging of the surface or to periodic alterations in the index of refraction of the tendon. Our reasoning can be outlined as follows:

1. if there were a periodic structure in the surface, it must be just as visible in the reflected-, as in the refracted light.

2. in the reflected light no diffraction-spectra are visible.

3. therefore, the looked-for periodicity is not due to a ridging of the surface.

Ad 1. Let us consider a grating, consisting of a homogeneous medium of an index of refraction n_1 bordering on a medium of an index of refraction n (Fig. 2). We confine ourselves to the simplest grating-profile $\zeta = c_1 \cos px + s_1 \sin px$, and we suppose that the incident pencil of rays is normal to the boundary plane. Now we compare the amplitudes of the flexure-spectra of the first order A_1 originating from reflection, with the amplitudes of the spectra B_1, occurring in the refracted light.[1])

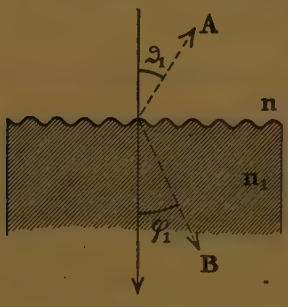


Fig. 2.

a. For polarized light, of which the electric vector is parallel to the grooves, $B_1 = A_1$.

b. For polarized light, of which the electric vector is perpendicular to the grooves, $B_1 = -A_1 \frac{\cos \vartheta_1 \, n}{\cos \varphi_1 \, n_1}$, in which ϑ_1 and φ_1 are the diffraction-angles for the reflected and for the refracted rays. In the preparations studied by us n and n_1 differ little, and the grating-period is so large that we can put $\vartheta_1 = \varphi_1$; so that B_1 is about A_1. So the diffraction-spectra are about as strong in the reflected as in the refracted light.

Ad 2. Now we try to see the diffraction bands in the reflected light. We look at our point-shaped light-source reflected in the preparation, while holding the object-glass towards the light-source, and the cover-glass away from the light-source. Then we see two images of the light-

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[1]) Cf. RAYLEIGH, On the dynamical theory of Gratings, Papers V, 403; Proc. R. Soc. 1907, A 79, 322.

source at a short distance apart when we look through a part of the object-glass that is slightly prismatic. The brightest image A is reflected by the anterior glass-surface. The less bright image B is reflected by the surface of the cover-glass, and has gone twice through the preparation. An image C reflected by the preparation itself, would nearly coincide with B. Now in order to isolate the image C, we press a piece of black paper on the surface of the cover-glass and take care to preserve optic contact by means of a drop of cedar-oil between the paper and the glass. The image B reflected by the posterior plane of the cover-glass must disappear now and only an image C could remain. Now from observation we learn that no trace is left either of a reflected image or of a diffraction-spectrum. This shows that the index of refraction of the preparation is so small that the reflected spectra have no appreciable intensity.

Ad. 3. Thus a periodic ridging of the surface cannot have contributed appreciably to the formation of diffraction-spectra observed in transmitted light.

On the other hand it is quite conceivable that these diffraction-spectra could originate from periodic differences of the index of refraction, since they exert an influence over the whole thickness of the preparation. Generally the diffraction-spectra must then be the more distinct, according as the preparation is thicker. This generally came true.

Now we still intend to demonstrate that the diffraction-spectra observed, are not at all to be ascribed to periodic changes in the index of refraction of an isotropic medium, i.e. to periodic variations in double refraction.

Let a polarizer be placed before the preparation. The diffraction-spectra will then be seen to retain their intensity whatever the position of this polarizer may be. Now if we arrange it parallel to the direction of the fibres, and if we place between the preparation and the eye an analyzer, which we rotate so that the light-source becomes lighter and darker every 90°, then the diffraction-spectra of the first order appear to exhibit simultaneous variations of intensity, but in an opposite sense. The spectra of the 2^{d} order (this observation is difficult of execution) seem to become clearer or darker as the lightsource becomes clearer or darker.

Now something remarkable can be witnessed, viz. that the light-source itself, which produces the diffraction-spectra is almost extinguished, while the first diffraction-spectra of that light-source still display an almost undiminished intensity. Such a phenomenon can never be explained by a periodic variation in the usual index of refraction, where as it is plausible that it is caused by variation in the double refraction.

We also examined our preparation under the microscope. It showed most distinctly with incident light, but also with transmitted light, a sinuous course of the tendon-bundles, as is illustrated, e.g. in Fig. 143 by SCHMIDT "Die Bausteine des Tierkörpers" (Bonn. 1924). What struck us here, is that the phases of undulation of the various contiguous bundles

are in complete agreement with each other. From this results a striation of the whole preparation. On the face of it this somewhat reminds us of the striation of the fibres of voluntary muscles. Direct measuring of the period of a group of clearly defined wavelets invariably resulted in the optically determined wavelength: 40 μ.

We now asked the question whether the phenomenon is of common occurrence. Nine preparations of the same cow's tendon, of thicknesses between 20 and 200 μ all displayed the same phenomenon, and the values of the periods varied from 38 μ to 41 μ. Afterwards we found the same in cross-sections of other cow's tendons and also of pig's tendons, and always with periods of about the same magnitude.

Between the optic diffraction-phenomenon and the periodic structure in the microscopic preparation there is no doubt a close relation. What this relation is we shall discuss in a later paper.

Further data were furnished by an inquiry under the polarization microscope.

Between crossed Nicol prisms the preparation is rather evenly clear, when the direction of the fibres makes an angle of 45° with the direction of the vibrations of the nicols. If the fibres are made to run parallel to the vibrations, we see alternately dark and light bands, normal to the fibres, with a distance of 20 μ between two dark (resp. clear) bands. Wherever the spiral of the bundles is seen to run parallel to the direction of the fibres, the transverse bands are dark, wherever the inclined pieces of the spiral are seen, they are light (fig. 3a). If the preparation is subsequently made to incline at a small angle, the dark bands are seen to merge into each other (fig. 3*b*), and the period of the dark bands is doubled (40 μ). It again comes true here that the dark bands are precisely where the direction of the spiral is parallel to that of the nicol.

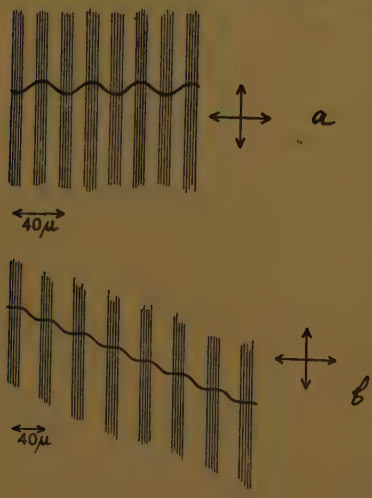

Fig. 3 *a* and *b*.

All these phenomena prove clearly that the bundles behave completely like uniaxial crystals, of which the axes are disposed in a regularly waving line, or on a spiral which we see in the projection. In connection with the polarization-microscopic investigations of AMBRONN, c.s., according to which the "micellae" that build up our tendon-substance, are to be regarded as unidirectional, uniaxial crystals, we can conclude that the micellae are also disposed either on a waving line, or on a spiral.

In order to choose between these two suppositions we made a number of cross-sections of one and the same cow's tendon, all parallel to the axis, but in two mutually vertical planes. The microscopic examination showed that in both preparations the bundles exhibited the same periodicity, and that in either of them the same flexure-spectra appear. It can, therefore, hardly be imagined that the preparation should consist of a large number of plane sinecurves, orientated in all sorts of azimuths, considering the complete contiguity of the bundles, observed in the preparation. Moreover, we see distinctly the spiral form, when tracing the boundary-line of a bundle with the screw of the micrometer. On this basis, then, the conclusion is warrantable that the observed sinecurves are, indeed, the projection of spirals, and that consequently the micellae are arranged in such spirals.

(From the Laboratory for Embryology and Histology and from the Physical Laboratory of the State-University at Utrecht.)

Utrecht, June 23, 1926.

Histology. — *"An Inquiry into the Physico-chemical Structure of the Collagenic Substance. III. Report on an Investigation with Röntgenrays of the structure of the Collagenic Substance"*. By Prof. G. C. HERINGA and N. H. KOLKMEIJER. (Communicated by Prof. J. BOEKE).

(Communicated at the meeting of June 26, 1926.)

1. Already a few years ago we took a photo of a tendon from a mouse-tail. The film was stretched along the wall of a cylindric camera with a radius of 2.72 cm.; the tendon was placed in the axis of the cylinder in paraffinoil, inside a thin-walled glass-tube, and the parallel bundle of monochromatic (CuKα) Röntgenrays was normal to the axis. The exposure lasted about 20 hours at about 10 mA and 40.000 Volts. Circumstances prevented us from prosecuting the examination until very recently. We have now taken a similar exposure, this time, however, without paraffin-oil in order to avoid the influence of this complication. The tendon, approximately $^1/_2$ mm. thick, was now dried, consequently stiff, and was thus placed in the thin-walled tube. During the exposure of six hours at 40000 volts, 15 mA. the preparation was rotated round the axis. Although the film is less blackened, this photo showed no appreciable difference with the previous one.

2. It is visible at once that the photo is of the nature of a four-point-diagram. This indicates a fibrous structure such as HERZOG [1]), and POLANYI [2]) and their co-workers found in fibrous substances. This has also been observed by KATZ [3]). On closer inspection it will be seen that the four points, which belong together, enlarge every time so as to form very broad and vague DEBYE-SCHERRER-lines. Still, two or perhaps four broad serial lines (Schichtlinien) remain visible, also fairly uninterrupted. We have also noticed a series of finer lines, parallel to the serial lines, much more difficult of observation. In order to ascertain the existence of these lines, we thought it necessary to make a photometric measurement of the film along some lines, normal to the direction of the striation. We are greatly indebted to Dr MINNAERT for performing this measurement for us. From the

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[1]) R. O. HERZOG and W. JANCKE, Z. f. Phys. **3**, 196, 1921.

[2]) M. POLANYI, Z. f. Phys. **7**, 149, 1921; M. POLANYI and K. WEISSENBERG, ibid **9**, 123, 1922 and following communications.

[3]) I. R. KATZ, Erg. der exact. Naturw. **3**, 316, 1924 and **4**, 154, 1925.

photometric curves it can be concluded that the above mentioned fine, alternately dark and light striation runs approximately parallel and straight, with equal breadths, over a large portion of the film. It may further be observed that this striation can be noted on the reproduction of the centre of an exposure of the tendon published by KATZ in Kolloïdzeitschrift June 1926.

3. The indistinctness of all the lines on both films is such that exact measurements are impossible. However, in order to verify a theory [1]) advanced by one of us (*H*) concerning the structure of collagenic tissue, it is very desirable to collect quantitative data. Therefore, we publish our results, be it provisionally and with some reserve, hoping that other observers will also publish theirs for the sake of comparison.

The median lines of some broad DEBIJE-SCHERRER-lines are to be seen at distances from the middle of the film, of 1.0, 2.0, 3.65, and 6.2 cm. from which we computed for $sin^2 {}^1/_2 \vartheta$ the values 0.033, 0.13, 0.39, and 0.825, in the roughly calculated ratios of 1 : 4 : 12 : 25. It should here be observed that the line 3.65 is very broad and probably consists of two lines. A more accurate estimation suggests that the number 12 of the last series should be replaced by 9 and 16. So we think that only one face of the crystal has reflected, which might point to a plate-shaped rather than to an elongated form of the crystallites. In case this hypothesis is correct $a = 4$ Å is to be considered as a possible parameter of the net.

From the formula given by DEBIJE and SCHERRER [2])

$$B = 2\sqrt{\frac{\log e}{\pi}}\frac{\lambda}{D}\frac{1}{\cos\vartheta}$$

we compute some tens of Å for the dimension of the crystallites with aid of the estimated breadth (to half the intensity of the middle) of the lines, viz. 2 mm for the line of the second order. For the lowest limit we found 14 Å.

From the distance of the distinct serial lines 5.9 mm we compute 87 degrees for the most frequently occurring angle, made by the one reflecting plane with the axis of the fibre. We believe that no conclusion may as yet be deduced from our exposures for the nature of the special fibre-structure in the sense K. WEISSENBERG takes it [3]). It seems to us that our measurements of the distances of two light lines in the fine striation by means of a photometric curve, is more accurate, viz. $\frac{2.2}{20.3}$ cm. We think that this striation is due to the occurence of a linear

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[1]) G. C. HERINGA and H. A. LOHR. These Proceedings p. 1081.

[2]) Nachr. Ges. Wiss., Göttingen, 91. 1918, Assuming the net to be cubic.

[3]) Zs. f. Physik, 8, 20, 1922.

succession of equal elements in the long axis of the fibres. If this is the case let the distance of the elements be x and let ϑ be the angle at which the distance between two light lines is seen from the preparation, then $sin\ \vartheta = \frac{\lambda}{x}$. In working this out the result will be $x = 0.004\ \mu$.

(From the Laboratory for Embryology and Histology and the VAN 'T HOFF-*Laboratory of the State University at Utrecht).*

Utrecht, 27 June 1926.

Petrology. — *"On a Young-Tertiairy Limestone of the Isle of Rotti with Coccoliths, Calci- and Manganese-peroxide-Spherulites".* By TAN SIN HOK. (Communicated by Prof. H. A. BROUWER).

(Communicated at the meeting of June 26, 1926)

Introduction. The in many respects remarkable collection of young-tertiairy limestones, collected by Professor Dr. H. A. BROUWER on the occasion of the Dutch Timor-Expedition, was kindly placed at my disposal for sedimentary-petrographical research.

In preparing this paper the author has been put under special obligation to Professor BROUWER for his extremely valuable directions and stimulative interest.

These limestones which are believed to be of pliocene age, are very frequent in the South of Rotti; they are found with reefs [1]). The groups diagenetically not much altered, were examined, all being white and soft. The following division could be made:

1. Radiolarian-limestones.
2. Foraminifera-limestones.

The groundmass of these limestones is very fine-grained; in group 2 aragonite-asterisks occur very numerously, whereas they are not yet met with in group 1. This paper is confined to an abnormal type out of the group of the foraminifera-limestones.

Method of Examination. Thin slices as well as preparations made of the disintegrated limestone, magnified about 375—750, were examined in monochromatic yellow light.

The monochromatic light was used to simplify the recognation of spherulitic and fibro-radiate forms. As is known fibro-radiate agregates of optically uni-axial crystals exhibit in parallel-polarized light the interferencefigure of homogeneous uni-axial crystals in convergent-polarized light [2]). Monochromatic light makes the interference-cross more conspicuous, so that those agregates cannot be overlooked so easily.

Making of the preparations. A fragment of the stone is boiled in distilled water. By this treatment it was completely disintegrated without any damage to its components.

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[1]) H. A. BROUWER **3**, p. 72.

[2]) H. ROSENBUSCH **16**, p. 398. G. LINCK **9**, p. 281.

Agregates of rhombic crystals of analogous structure form interference-figures of little difference.

A drop of the resulting milk of lime was evaporated on a slide, and on the residue some diluted canada-balsam was brought (as a solvent xylol was used).

Professor S. SQUINABOL of Turin recommended this diluted balsam for the mounting of preparations of Radiolaria. In this soft medium they can be brought to rotation by a gentle pressure on the cover-slip, so that now they may be studied in every position.

This diluted balsam can be generally recommended for sedimentary-petrographical researches.

For the concentration of heavier or coarser ingredients the method which is already known, was followed: the beaker with the milk of lime was brought in a rotatory motion, the desired concentrate accumulates in the least agitated parts of the fluid and can be transferred by a pipette.

Description of specimen: 168 ×, from Bebalain. Rotti.

Macroscopically a gray, dirty-white rock, soft; coming-off, with many spots of MnO_2, often surrounded by a brown envelope.

Microscopically a compact limestone, fine-grained;

A. the coarsest constituents of which are:

a. Foraminifera which reach a maximum dimension of 100 μ.

Globigerinidae and fragments of larger globigerina.

Textularidae.

The planctonic foraminifera predominate, very curious is *that larger individuals are absent.*

b. Terrigenous fragments of lime: they are of the same order of size as the foraminifera, usually with sharp edges, therefore not distinctly marked off against the groundmass, many of them do not extinct homogeneously.

Mineralogically they consist of aragonite or calcite, (discernment by means of MOHR's salt: $Fe\,SO_4\,(NH_4)2\,SO_4\,6\,H_2O$ [1]).

c. volcanic glass, subordinate, occurring in transparent pieces with many opaque spots; between crossed nicols, minerals with gray polarization-colours, probable feldspars, are visible.

d. concretions of manganese-peroxide, munerous, as little spots or cloudy accumulations of irregular shape. The largest accumulation in the sections is 240×135 μ. Where the grains are not closely accumulated a brown colour is often visible. The manganese-peroxide encloses the ground-mass and fills also the chambers of the globigerina.

Qualitatively the manganese was analysed by dissolving the grains in a bead of salt of phosporus and then performing the colourreaction by oxidation with KNO_3.

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[1]) K. KEILHACK, 7, p. 397. For the examination of preparations it is easier to apply this reaction than MEIGEN's, and moreover, it produces still distinct colours with objects of minute dimensions.

The reactions from which the conclusion was drawn that the manganese occurs as superoxide were:

1. evolution of Cl_2 from concentrated HCl.
2. evolution of O_2 from H_2O_2.

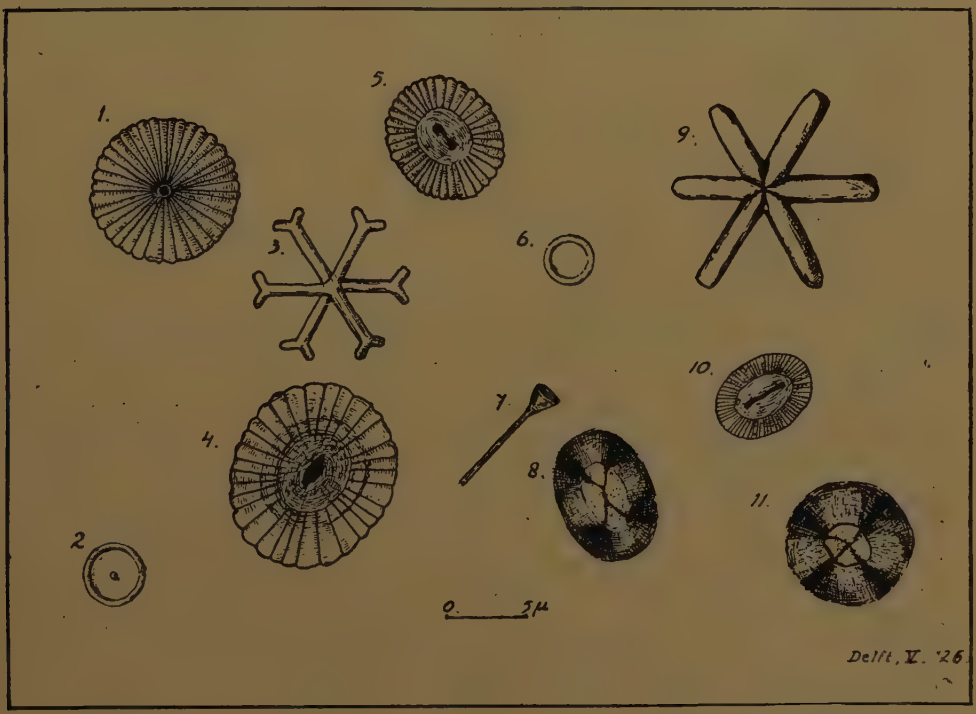
These reactions were performed in the laboratory of microbiology by Ir. C. B. VAN NIEL t. to whom the author feels greatly obliged.

B. The finer components of the ground-mass are:

e. Coccoliths;
f. Asterisks ("Disco-asters");
g. Calci-spherulites;
h. Grains of calcite;
i. Grains of aragonite.

e. Coccolithophoridae Lohmann [1])

circular, oval to elliptical disks, diameter 2—20 μ, most of them are about 10 μ and less. Thickness 3 μ and less.



Discoliths: fig. 2, 6.
Placoliths: fig. 1, 4, 5, 8, 10, 11 (8 and 11 are respectively 5 and 1 between crossed nicols).
Rhabdolith: fig. 7.
Disco-asters: fig. 3, 9.

Magnified 2000 ×.

[1]) H. LOHMANN **10**, p. 147; J. SCHILLER **17**, p. 282.

Discoliths fig. 2, 6.

not perforated, diameter till 4 μ, thickness about 1 μ, margin thickened. Sometimes with a button in the center fig. 2.

They are easily to be brought to the system of LOHMANN [1]).

The most frequent of these have been pictured:

Pontosphaera huxleyi Lohmann fig. 6.

Pontosphaera pellucida Lohmann fig. 2.

Placoliths (Lohmann).

with singular perforation, fig. 1, 4.

with double perforation, fig. 5.

with irregular perforation, fig. 10.

The surface is striated radially [2]). LOHMANN's conception (loc. cit. page 114) that the margin of the placoliths is thinner than the center, was confirmed.

By rotating of the placoliths, the differentiation in basal and distal disk could be observed, these disks appear to differ in size, the plane of the disks is convex-concave.

The figures given in the sketch belong to the Coccolithophora Lohmann and are identified as:

Coccolithophora leptophora Murray and Blackman fig. 1.

Coccolithophora pelagica (Wallich) Lohmann fig. 4, 5, 10.

Rhabdoliths: It is remarkable that they occur in a very subordinate way.

Fig. 7: Discosphaera Thomisonii Ostenfeld with a distinct axial perforation.

Structure of the Coccoliths: They consist of radially placed needles which diverge in the placoliths from the perforation or perforations; these fibres cause the radial striations on the disk, an adoption of radiating channels within the disk for their explication is superfluous [3]).

The optical orientation of the fibres is negative. Which of the other axes of elasticity lies in the disk-plane could not be determined. In the disk-plane the fibres are not placed perfectly radially, eventually caused by ramification of the fibres. This appears from the behaviour of the interference-figure when revolving the microscopic stage. Then the arms of the cross do not keep the same orientation. Fig. 11 is a Coccolithophora leptophora Murr. and Blackm., fig. 8 is a C. pelagica (Wallich) Lohm. between crossed nicols. In consequence of the structure of 2 disks the interference-figure of the disk-center often differs from that of the margin.

It is very probable that in space the aragonite-fibres do not diverge from one single center, as the spherulites, but from various centers of the disk-axis. In vertical section the extinction is not homogeneous [4]). By

[1]) H. LOHMANN **10**, p. 147; J. SCHILLER **17**, p. 282.

[2]) These striations are only visible on the largest disks.

[3]) A. VOELLTZKOW **21**, p. 483.

[4]) This observation could only be made on the bigger placoliths.

optical way with the method of SCHROEDER VAN DER KOLK the mineralogical modification of $CaCO_3$ which form the coccoliths, could not be determined. The colouring with MOHR's salt, however, was positive for aragonite.

Nor do the rhaboliths extinct homogeneously in profile, so that probably they too are built up of fibres which in space are arranged analogically as in the coccoliths.

f. Asterisks. ("Disco-asters") fig. 3, 9.

These too consist of aragonite; to which organisms they belonged, cannot yet be decided. They will be treated in another paper.

g. Calci-spherulites.

Radiated in structure, irregular in external form, size 2—5 μ. First they were thought to be coccoliths, but they differ from these in:

1. compared with coccoliths of the same size they exhibit a higher interference-colour;

2. when rotated round an axis in a plane parallel to the stage, they appear to be no disks at all, the interference-cross *remains visible in all positions*;

3. in structure they are perfectly spherulitic, the interference-figure persist with the same orientation of the cross-arms as well during the revolution of the stage as on being revolved on an axis in the plane of the stage.

In general one can expect that the structure of crystalline products of chemical or biochemical extraplasmatic origin, especially, in the beginning of their formation, follows as accurately as possible the laws of crystallography, whereas the products of physiological-intraplasmatic origin in the first place aims at reaching a definite morphological ultimate shape.

The discernment between spherulites and coccoliths does not always run easily, they also consist of aragonite-needles, with negative orientation; perhaps here we have to do with the modification Vaterite [1]).

h. Grains of calcite.

By means of colouring with MOHR's salt they could be recognised and distinguished from:

i. Grains of aragonite,

also occurring in irregular forms, which often do not extinct homogeneously.

Remark: After dissolving the rock in respectively HNO_3, HCl, acetic acid a residue was obtained, which absorbs organic colouring matters, obviously the lime contains some clay.

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[1]) F. KLOCKMANN 8, p. 455 and p. 457. Below 29° C., from solutions of calcium-carbonate, calcite will always precipitate. Above this temperature aragonite, and in the presence of free bases as ammonia the spherulitic Vaterite. Aragonite, however, may also be formed below 29° C. especially in the presence of Magnesia-salts, e.g. in seawater.

Views on the various components of the rock.

To avoid repetition in the following statements the premise is made that in this rock diagenetic alteration, if it has taken place, must be very subordinate.

Phenomena of solution were not observed on the aragonite-components, whereas their minute dimensions and the porosity of the rock had made them highly accessible for chemical attack; if solutions had circulated in this rock those particals certainly would have acted as crystallization-centers, and so would have furthered a precipitation, but the fact that the disintegration of the sample was so easily done, as was previously stated, makes it evident that any posterior impregnation by solutions, saturated with calcium- or maganese-carbonates must be excluded.

While the tropical climate favours the chemical weathering the elevation of this rock must be of very recent date.

The occurrence with coralreefs points at a sedimentation in a shallow sea, notwithstanding the pelagic habit of the rock.

Concretions of manganese: MOLENGRAAFF[1]) in 1915 came to the following conclusion: "Nodules and concretions of manganese in general (therefore) are not characteristic of abysmal deposits in this way that from the occurrence of such concretions in a certain deposit, one would be justified in concluding that the deposit could be nothing else than an abysmal deposit...."

The chemical process of their growth is a very slow one, and the chance of finding them is inversely proportional to the rate of accumulation. About the velocity of the sedimentation of our rock, no data can be obtained of course, but it is certain that it was greater than of abysmal sediments.

In the discussion following the communication of his paper, MOLENGRAAFF on WICHMANN's remark, admits the possibility of the formation of manganese-nodules by biochemical processes, but at the same time he points out that until now the existence of bacterial life in abysmal depths has not been proved (pag. 428).

At this moment manganese-bacteria from the sea or from sea-mud, not to mention those from abysmal depths have not been described.

BEIJERINCK[2]) demonstrated them in garden-mould, VON WOLZOGEN KÜHR[3]) found them as hydrobios in dune-water and it is not improbable that they are represented in the halobios, a supposition which Prof. Dr. A. J. KLUYVER thinks very probable (verbal communication).

In this connection the attention is drawn to the following quotation:

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[1]) G. A. F. MOLENGRAAFF **12**, p. 418.

[2]) M. W. BEIJERINCK **2**, p. 123.

[3]) C. A. H. VON WOLZOGEN KÜHR **21**, No. 3, 4, 5,

"Vielleicht wird man aber annehmen, dasz die Manganknollen ähnlich wie das Sumpferz durch Vermittelung von Bakterien ausgeschieden wurden, und dasz sie sich dort in gröszeren Mengen anhaufen konnten, wo die Lebensbedingungen für diese besonders günstig waren" [1].

Without further commentary BEIJERINCK (on page 128) calls the grains of manganese-superoxide formed by bacteria: spherulites, they are perfectly round and have a rough surface, and, besides, they should include organic matter. They reach sizes to 350 μ [2].

Another possible way of formations is chemical oxidation of the mangano-ion dissolved in the seawater, especially as it is known that seawater reacts alkaline and that the oxidation of mangano-ions depends on the concentration of the hydrogen-ions in the medium.

But, if this process had taken place, the Mn-spherulites would have been arranged more or less in layers, and would not occur as dispersed spots. Post-genetical formation must be excluded for reasons mentioned above. The only possible origin of the accumulations in this rock is that they are a product of bacteria, which lived in the superior parts of the sedimentated mud.

Then the accumulations ("star-accumulations") might be considered as the products of secretion of the former colonies.

MOLENGRAAFF's pronouncement: "Consequently concretions of manganese are in this manner characteristic of abysmal deposits that they may form an important percentage in proportion to other constituents exclusively in such deposits", now can be stated precisely in this manner that the manganese-superoxide as: "important percentage in proportion to the other constituents" is only in this case an indication of an abysmal sediment, when it occurs in big grains: *non numerus granulorum sed magnitudo grani.* For, the quantity is as well a function of the favourableness of biological factors, whereas the magnitude, it being of no consequence whether the precipitation was caused by chemical or biochemical processes, in the first place, is ruled by the time available for *undisturbed* growth, thus is connected with the velocity of sedimentation of the other constituents of the rock; circumstances which, without any doubt, are as favourable as possible in abysmal depths.

Calci-spherulites. STEINMANN [3] asserts that the coccoliths occurring in biancone, lay in a ground-mass consisting of grains which should have a fibrous-crystalline structure. He asserts them to be "Verwesungsfällungskalk". Besides from the decay of organic matter, calci-spherulites can be obtained in the laboratory in a purely anorganic way. (HARTING [4])

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[1] J. MURRAY and E. PHILIPPI **14**, p. 190.

[2] On the analogy of BEIJERINCK's nomenclature the MnO_2-grains in this rock, ar called spherulites.

[3] G. STEINMANN **18**, p. 444—445.

[4] P. HARTING **5**, p. 1—84.

and see quotation on page 1099) Physical-chemical precipitations of lime are not impossible in nature [1]).

It cannot be decided in which of these ways the calci-spherulites in this rock have been formed. A sure indication of the biochemical formation is the occurrence of organisms, whereas there are no terms to consider a co-existing physical-chemical "planktogeneous" formation as impossible.

ANDREE [2]) makes a difference between "Verwesungsfällungskalk" viz. lime issued from precipitation in sea-water by the action of "decay"-products and "physiologische Fällungskalk" viz. lime, which is formed in direct connection with the physiology of living organism, in the first place of bacteria.

Other geologists too, accept on DREW's authority that there exist bacteria which may be brought to a physiological group of: "Calcium-carbonate-bacteria", on the ground of the fact that the production of calcium-carbonate is to be reckoned among the direct physiological processes of those bacteria.

However, it ought to be mentioned, that evidently this conception is not agreed with in biological cercles. So the well known bacteriologist MOLISCH gives the following definition of "calcium-carbonate-bacteria": "Die Bacterien erzeugen entweder aus Eiweisz, ihren Derivaten oder Nitraten Ammoniak, dieses verbindet sich mit der im Wasser gelösten Kalksalzen ..... zu Kohlensauren Kalk, der entweder für sich allein oder mit Phosphorsäure zusammen in Form der Sphärite erscheint [3]).

It is evident that the difference between: "Verwesungsfällungskalk" and "physiologische Fällungskalk" mentioned above, *has no reason to be kept*, a conclusion backed by Professor Dr. A. J. KLUYVER.

Bacteria which produce lime were met with by DREW in great numbers in the surface-water and in the mud of the sea of Florida, they only prosper in warm seas and in the most florid way in depths less than 100—200 fathoms [4]).

In the mud in loco VAUGHAN [5]) found calci-spherulites of 4—6 μ. HARTING's and STEINMANN's [6]) experiments might cause the supposition that the formation of spherulites, is only possible in a viscous medium. Professor Dr. A. J. KLUYVER kindly communicated to me: "A viscous medium is *certainly* not necessary for the formation of calci-spherulites, *perhaps not* for the manganese-spherulites.

A further indication of the conditions during the sedimentation of this rock gives the following quotation from MOLISCH's paper (loc. cit. page 135): "Die Menge des Ammoniaks musz eine gewisse Höhe erreichen,

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[1]) ARN. HEIM **6**, p. 30—32.
[2]) K. ANDREE **1**, p. 185—191.
[3]) H. MOLISCH **11**, p. 130—139.
[4]) H. DREW **4**, p. 35.
[5]) F. W. VAUGHAN **20**, p. 53.
[6]) G. STEINMANN **19** p. 40—45.

wenn die Fällung des CO_3 Ca erfolgen soll". Strong oceanic currents are consequently not to be expected.

HARTING, STEINMANN [1]) and BEIJERINCK (loc. cit.) found in the spherulites from their experiments an organic matter with proporties of conchyoline. It is possible that part of the residue, obtained by dissolving the rock, and which was considered as clay, may be conchyoline.

Coccolithophoridae [2]) are uni-cellular Flagellata which belong to the nanno-phyto-plankton. They avoid sweetened sea-water and are met with in tropical and temperates seas. According to MURRAY and RENARD Rhabdosphaera and Discosphaera Lohm. would have their greatest development in aequatorial seas, Coccolithophora Lohm. in the seas of the temperate zones.

Curious is that in this sediment, though certainly formed in the tropics Rhabdosphaera and Discosphaera occur very subordinately.

The origin of the rock: [3]) About the young-tertiairy Globigerina-rocks of Rotti, BROUWER (loc. cit. page 76) writes: ... "achten wij het waarschijnlijk dat ze — hoewel in samenstelling veel overeenkomst vertoonend met sommige afzettingen van recent globigerinenslib in diepzee — niet op zeer groote diepte, maar in een zee, waarin misschien koraaleilanden aanwezig waren, zijn afgezet. Wij kunnen b.v. denken aan soortgelijke afzettingen als de kalkslibvormingen in lagunen van koraalriffen en die welke ontstaan uit met koraalmelk beladen, „witte water", dat na stormen tot verscheidene kilometers afstand van koraalriffen voorkomt, waaruit het kalkslib gelijktijdig met de globigerinenschalen is bezonken.

Of, de globigerinengesteenten kunnen zijn afgezet op een dicht onder de zeeoppervlakte gelegen rug, die door diepere zeeën met gunstige levensvoorwaarden voor planktonische foraminiferen was omgeven" [4])

In literature a recent mud of the composition of this rock is not known. Of the fossil limestones the limes of Albrada, a raised atoll in the Indian Ocean [5]) are to be compared with this rock. But VOELTZKOW

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[1]) G. STEINMANN 19, p. 40—45.

[2]) H. LOHMANN loc. cit. J. SCHILLER loc. cit.

[3]) In the microbiological parts the author's thanks are due to Prof. Dr. A. J. KLUYVER and Ir. C. B. VAN NIEL, to whose better judgment the questions were submitted.

[4]) "... we think it probable that — in spite of the great resemblance of their composition with some sediments of recent globerina-ooze in the deepsea — they were not sedimentated in very greath depth, but in a sea, where perhaps coral-islands were present.

For instance, we can think of analogous sediments as the formations of calcareous-mud in lagoons of coral-reefs and those which issue from "white-water" loaded with coral-milk that after storms occurs up to several kilometers' distance from coral-reefs, from which the calcareous-mud simultaneously with the shells of the globerina were sedimentated.

Either, the globigerina-rocks were formed on a ridge situated close to the sea-surface and surrounded by deeper seas with favourable conditions of life for planktonic foraminifera".

[5]) A. VOELTZKOW 21.

still owes an explanation for the occurrence of the coccoliths in "pure culture".

BROUWER's first explication is most probable for this rock, but under very circumstances.

The geological occurrence in connection with coral-reefs is an indication of a formation in shallow water: the fine-granularity, the occurrence of calci-spherulites for a quiet medium, circumstances which are to be found in the *most quiet* parts of the lagoon of an atoll, far from the entrances to the open ocean and which in this special case were met with far from the shores of the lagoon. So the rock represents the finest "washings" of coral-mud, brought by weak currents, stronger sea-currents which could transport coarser material do not occur or are very scarce. The accumulated coral-mud consisted of fine detrital lime issued from the atoll, of coccospheres and little foraminifera. They could settle in this part of the lagoon, the organisms died, the dead albumen-matter were putrefied by bacteria, the issuing ammonia reached a concentration necessary for the precipitation of calcium-carbonate, besides, the manganese-compounds in the volcanic ash were transformed in manganous-carbonate. In this part of the lagoon probably only bacterial life was possible, all organisms occurring in this rock must be originated from individuals not living in situ, a conclusion drawn from the absence of greater foraminifera. [1])

The organisms, bearers of the "Disco-asters" must not be considered as a form adapted to these special conditions, they are common in the group of the soft globigerina-rocks, which represent more normal coral-mud: (e.g. 178 × from Bebalain).

This rock-abnormity can be nothing but a local facies.

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[1]) For the "azoism" the denitifying bacteria could be hold responsible (BRANDT's hypothesis). The flora and consequently the fauna too, due to the resulting scanty percentage, eventually, the total absence of nitrates.

The circumstance that the waterrefreshment is very defective makes comprehensible that the bacteria consume the nitrates almost completely.

It is possible too that this part of the lagoon runs dry when the tide is low, but for this I have no indication. (e.g. mud-cracks).

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Laboratory of Stratigraphical Geology and Palaeontology.

Delft, June 1926.

Physics. — *"Influence of the Pressure on the Electric Conductivity of Platinum."* (15th communication of results obtained by the aid of the VAN DER WAALS Fund.) By A. MICHELS and P. GEELS. (Communicated by Prof. J. D. VAN DER WAALS Jr.)

(Communicated at the meeting of May 29. 1926).

The desire to improve the temperature measurements in the determinations of isotherms induced us to make new measurements of the influence of the pressure on the electric conductivity of platinum.

In the earlier measurements of isotherms a doubt had risen on the question whether the temperature within the piezometer in which the gas to be examined was compressed, was really the same as that of the thermostat surrounding the piezometer. This doubt was founded on some small deviations in the results of the measurements. A motive for the suspicion that we should have to do here with a difference of the two temperatures, was obvious.

In the earlier arrangement, only that part of the piezometer and its protecting casing jacket reached into the thermostat, where the capillary form of the piezometer began, and the measurements actually were made. The remaining part, which represents a considerable weight of steel and mercury, remains outside this thermostat. As the mercury in this reservoir forms a continuous column with the mercury in the capillary and the much greater quantum of the reservoir is permanently in contact with the outer air, it was quite possible that a continuous transmission of heat took place along this mercury column. Moreover along the inner side of the steel protecting jacket a conduction of heat would be able to prevent a heat equilibrium from ever being established.

A preliminary determination, made in the most favourable case for reaching equality of temperature between compressed gas and thermostat with difference of temperature with the outer air of about 40 degrees gave deviations of $^1/_{50}$ degree already.

These observations were arranged as follows (fig. 1): the head of an old measuring capillary was blown into a bulb and then opened at the top (*a*), after which the capillary was placed in the usual way in its protecting jacket, with this difference, however, that the upper closure of this jacket had been removed. Then the reservoir of a BECKMANN thermometer was stuck into the bulb. Now the mercury in the capillary is just made to rise so as not to touch the thermometer. The thermostat, which surrounds the protecting tube in the usual way (*b*) is brought to the desired temperature, and the difference between this temperature and that of the gas in the bulb is measured. The results were as given above, and this, too, in the

case in which the mercury stood as high as possible in the capillary. At a lower level we should certainly have to expect greater differences.

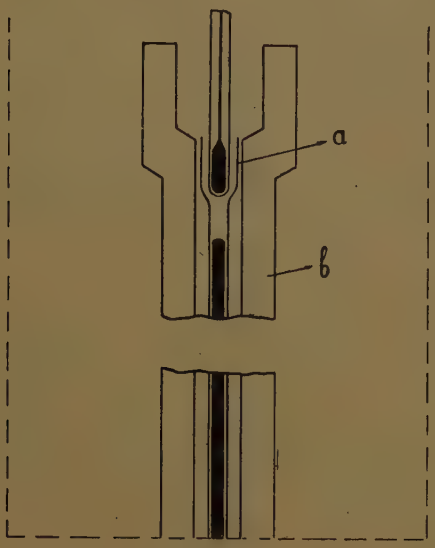

Fig. 1.

Though it is obvious that the above-mentioned measurements can give an insight into the existence and the order of magnitude of the errors made in the isotherm determinations, no accurate determination of errors could be expected in this way. For this purpose it would be necessary to determine the temperature within the piezometer *during* the pressure measurements. If this is possible, there are still two ways open for isotherm measurements. Either keeping the same instruments and using the internal temperature as measured in the calculations, or examining by the aid of an interior thermometer in order the instruments have to be modified to reach a sufficient equality of the inner and outer temperature. Which solution is to be preferred does not enter into the scope of this communication.

A second motive for the construction of an interior thermometer was the desire to know how much time adiabatic variation of temperature (consequence of variations of pressure) required to be absorbed in the thermostat and to know how rapidly the inner temperature follows the fluctuations of the temperature of the thermostat, which fluctuations can never be entirely avoided.

For the measurement of the inner temperature it seemed to us the most practical course to make use of a *Pt*-resistance thermometer. In this we are, however, at once up against the difficulty that the pressure also exercises an influence on the resistance. As regards the extent of this influence there exist only a few, and what is worse discrepant data. The authors who furnish numerical data are E. LISELL [1]), M. A. LAFAY [2]) and P. W. BRIDGMAN [3]). The first states that he has found

$$\alpha = 1{,}77 \times 10^{-6} + 3{,}85 \times 10^{-11}\, p$$

(translated by us into KG. per cm^3. from atm.), whereas LAFAY gives

$$\alpha = 1{,}80 \times 10^{-6} \text{ in KG. per cm}^2$$

and BRIDGMAN:

$\alpha = 1{,}90 \times 10^{-6}$ to 1000 atm. and at 12000 atm. $1{,}77 \times 10^{-6}$.

The last observer, who draws attention to the lack of harmony between

1) Thesis Upsala 1902.
2) Ann. de Chim. et de Phys. **19** (1910) 289.
3) Proc. Am. Acad. **52** (1917) 573.

the two first, reduces LISELL's values from atm. to kg. cm^2., whereas he does not do the same with LAFAY's figures. Thus he overrates the difference considerably. It must be admitted that LAFAY's communication on this subject leaves something to be desired as regards clearness.

It does not seem desirable to base a thermometer on the above-mentioned data. Besides, since the improvements in the method of the measuring of pressure we were able to work with more accuracy and to express our data in absolute units. These two considerations led us to a new determination of the influence of the pressure on the *Pt*-resistance, at first, however, only to 250 atm., as we required these results first of all.

Fig. 2.

Method of Measurement.

A preliminary calculation[1]) gave as most favourable dimension of the wire one of about 0.2 mm. with a length of from 5 to 6 m. This wire may be charged with a current of 0.004 amp. without danger of the temperature rising higher than 0.001 degree. The current as used in the experiment always remained below this value.

First this wire was heated for 10 min. at about 400°, and then wound bifilarly round a glass insulator (fig. 2) and suspended from a steel plate *B*. Through this steel plate four insulated transits were made. In our previous measurements the insulations were always made of ivory; here, however, ivory appeared not to possess a sufficient insulating power, especially at somewhat higher temperature. After having tried other insulating substances, amber at last proved successful. The insulation resistance lay mostly between 100 and 2000 mega ohm. In order to render a potentiometric method of measurement possible, four transits were applied. The two pairs of copper cores are connected bij a *pt* wire 1 mm. thick at the lower side of the plate. The wire which has to be examined, is welded on to this *pt* wire. By this arrangement all the contacts are brought into the pressure body and close together. This diminishes the chance of thermophenomena and of the influence of a onesided compression strain at the places of contact.

After the thread has been wound, it is once more gently heated in an oven to about 150 or 200° in order to eliminate possible tensions that have arisen through the winding and to prevent later thermal after effects.

The glass insulator with the *pt*-wire is placed in a tube-shaped pressure reservoir, which is closed at the top by plate *b*. A swivel presses *b* on the upper side of the

[1]) A. MICHELS. Thesis, Amsterdam, 1924.

reservoir and this, together with a packing ring, brings about a closure free from leakage. The whole reservoir is further filled with oil, which is carefully examined as regards its insulating power.

As it was our intention to measure the influences of the pressure to an accuracy corresponding to a temperature determination accurate to a 0.001 degree, it was necessary to take certain precautions, which exclude differences of temperature of this order. For this purpose a second platinum wire as far as possible of the same thickness, length, and suspension was put in to the thermostat in which the pressure reservoir was placed. This wire was, however, never exposed to pressure.

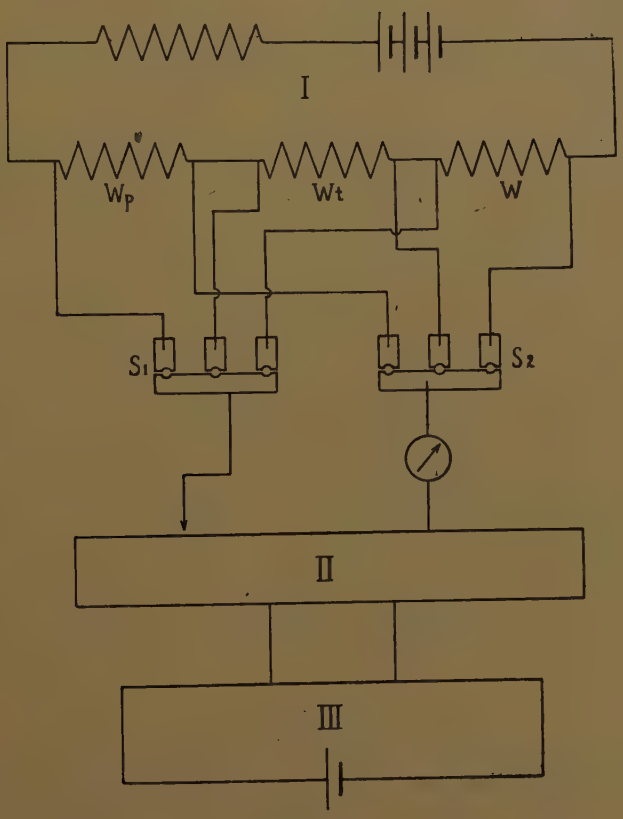

Fig. 3.

Before and after each measurement of the wire under pressure the second wire was measured. (Each series of measurements consisted of 3 such series and extends, therefore, in all over 9 measurements. These measurements took about 10 minutes.) It is possible to derive the required temperature correction for the first resistance from the values obtained from the second. In this of course should be supposed that the two thermometers possess the same inertia. This can, however, be accomplished by also placing the second wire in a casing of about the same thickness.

The electric method measurement consisted in a potentiometric replacement method. The arrangement is schematically represented in fig. 3. Here Wp is the first, Wt the second wire, W being an accurately gauged resistance box. S_1 and S_2 are switches, which enable us to connect Wp, Wt, and W successively to the potentiometer circuit II. The current for this circuit is provided by the shunt circuit III.

In the measurement of e.g. Wp the switches are adjusted in such a way that this resistance is switched on to the potentiometer, and the latter is adjusted to a minimum deviation of the galvanometer. Then W is coupled on the potentiometer, and the resistance, which is equivalent to Wp is determined. If W is not adjusted beforehand at about the right value, so that we have to modify it, the current in the resistance circuit changes a little in consequence; hence a second measurement becomes necessary for Wp.

By a suitable choice of the different resistances the whole operation can, however, be made exceedingly simple, and when e.g. beforehand W has been fixed at 0.01 Ohm, the whole measurement can be confined to a reading of the small galvanometer deviations, the fixing of the scalar value, and the changes in the switches S_1 and S_2.

To lessen the thermo-couples the whole complex of resistances was placed in a separate isolated space, and all the supply leads and contact knobs were made of the same copper.

The results obtained are given in the subjoined table. The pressure is given in kg. per cm^2. Under each pressure is found the total decrease of resistance per Ohm of initial resistance at the temperature in question. A second table states the mean decrease of resistance per kg. between every two successive observations. In addition figure 4 gives the graphical course.

All the results show a great difference from the data published up to now, in so far as the pressure coefficients vary more considerably with the pressure.

$$\frac{\Delta w \times 10^6}{w}$$

Pressure in kg. per cm^2.	51.5	101.5	151.5	201.5	251.5
15.57°	185	340	465	568	661
21.95°	191	351	484^5	602	708^5
34.75°	196	364^5	510	629	739

Besides, at the lower pressures the coefficient is to a much greater extent variable with the temperature.

These differences can be accounted for. All earlier observers have worked

$$\frac{\Delta w \times 10^6}{w \Delta p}$$

Pressure range in kg. per cm².	0—51.5	51.5—101.5	101.5—151.5	151.5—201.5	201.5—251.5
15.57°	3.68	3.11	2.50	2.05	1.86
21.95°	3.70⁵	3.20	2.67	2.25	2.03
34.75°	3.80⁵	3.37	2.92	2.37	2.20

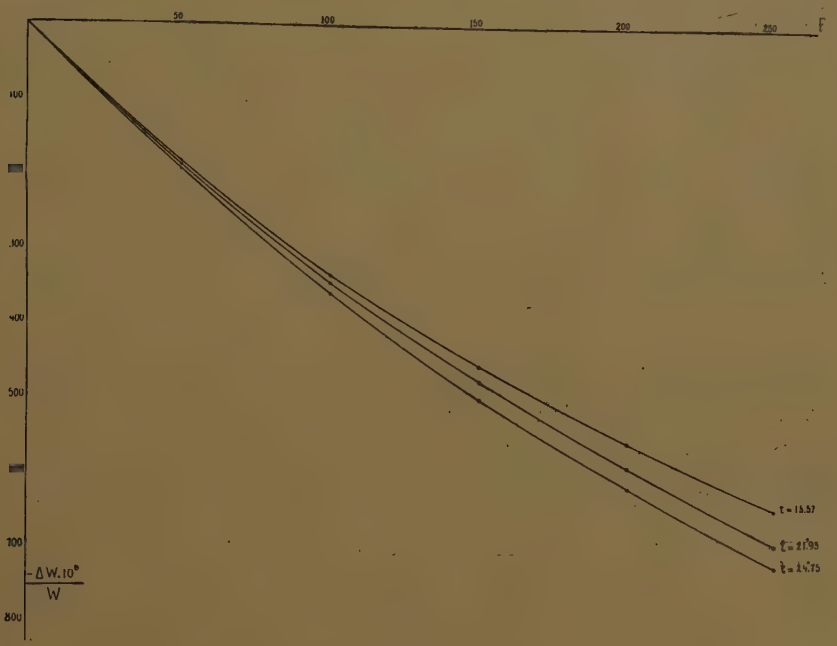

Fig. 4.

with much greater pressure intervals. Thus BRIDGMAN's first interval is 1000 atm. Nor does LISELL mention any observations below this pressure. Of LAFAY this cannot be ascertained, since he mentions only interpolation values ascending with 500 atm.

Now it may be very well possible that if these observers had worked with lower pressures, they would have found values more in harmony with ours. At higher pressures our values show a tendency to agree more with the coefficients known. In how far they are really in harmony with them, cannot yet be decided, because on the data given no extrapolation is possible up to e.g. 1000 atm.

If they actually agreed, it would also appear that a development of the resistance into a function

$$W = W_0 + \alpha p + \beta p^2$$

as has been tried among others by LISELL, is inadmissible. For a higher pressure the resistance shows a tendency towards an asymptotic course, whereas according to the equation the quadratic term would begin to dominate. This is the ground, why we have not tried to draw up a function on the values that we have now at our disposal. Small differences might still be accounted for by the greater accuracy with which we might measure our pressures, while a third cause may be found in a possible difference in the purity of the metals, the *Pt* used by BRIDGMAN and that employed by us not being supplied by the same purveyor, BRIDGMAN, too, was of opinion that the difference between his values and those of LISELL was due to a difference in purity. He supposed LISELL's material to contain 0.28 % of Iridium.

This, however, gives differences of only a small percentage, so that here too not so much influence may be assigned to this as to account for the whole difference. At first, when we did not yet suspect such a great difference in the observations, it was only our intention to construct a thermometer which was able to resist pressure. Hence no attempt was made to obtain material from the same source from which the other observers obtained theirs. Our observations enable us fully to fulfil our first intention. It has, however, also proved necessary to determine anew the influence of pressure on the conductivity of metals with smaller pressure intervals than those with which earlier observers worked. The preparations herefore are already in a pretty advanced stage. Measures have also been taken which we hope will allow us to attain greater accuracy in our electric measurements, in order to be also able to profit by the greater accuracy of our pressure measurements. Of course the greatest care shall be taken to obtain material of purity as high as possible.

Botany. — *"Concerning the sensibility of decapitated coleoptiles of Avena sativa for light and gravitation."* By H. E. DOLK. (Communicated by Prof. F. A. F. C. WENT.)

(Communicated at the meeting of May 29, 1926).

As the conceptions of the various investigators about the influence of the tip of the coleoptiles of Avena on the phototropical and geotropical curvatures differ in some respects (see the historical review of NIELSEN (3)) I thought, it would be interesting to investigate this question further.

For this purpose I used the observations of ROTHERT (5) about the regeneration of "a new physiological tip" in decapitated seedlings. In this paper I purpose to give an account of the results of my experiments.

By the removal of the tip the uppermost zone assumes the properties of the tip, which it did not possess before. ROTHERT has shown this for phototropical sensibility. These experiments were repeated and even extended to geotropism.

The question is, whether the transmission of the stimulus is generated by the formation of specific "stimulus substances" (STARK 7) or by a change of the concentration of the growth accelerating substances already present, (PAÁL (4) and others). SÖDING (6) has observed that some time after the decapitation the uppermost zone of the coleoptile again produces growth accelerating substances. If there is an immediate connection between the transmission of the phototropical stimulus and the production of the growth accelerating substances, then the phototropical sensibility has to return at exactly the same time as the growth acceleration in decapitated coleoptiles.

In order to ascertain this one ought to examine accurately:

1. The moment when the growth increases after the decapitation.
2. Whether the growth-accelerating is caused by the growth-accelerating substances or attributable to a temporary influence of the wound.
3. The moment when the phototropical sensibility returns.

Before measuring the growth I made some experiments in order to determine the influence of a regenerated tip on the growth of a stump. For this purpose the seedlings were decapitated ($\pm$ 4 mm). After twelve hours the new physiological tip is regenerated. Two millimeter of this tip were then cut off and placed unilaterally on the coleoptile stump. Seventeen out of 23 plants curved negatively, 2 positively and 4 remained straight. (The curvature is positive when the plant curves towards the side on which the ring is placed.) This experiment shows that the new regenerated tip produces growth accelerating substances. If the uppermost

coleoptile ring was cut off immediately after the decapitation and placed unilaterally on the stump, then the stumps only curved slightly. NIELSEN (3) also observed slight curvatures in this case.

The growth of decaptitated plants was measured and compared with the growth of the plants on which the tip was replaced. It was therefore necessary to examine whether the tip retains its normal properties under these circumstances. The plants were decapitated and the tips replaced on the stumps. After twelve hours the tips were removed and placed unilaterally on the same stumps. Out of 52 plants 5 curved positively, 5 negatively, while 42 remained straight. Some control experiments were made to examine whether the stumps were still able to curve. All the stumps curved very strongly, when tips, that were just cut off, were placed unilaterally on them.

Therefore we may conclude that a tip, placed on a stump, loses its activity after some time.

Is there any regeneration of the tip in a stump on which a tip is replaced ? When the tip was placed on the stump for 6 hours no regeneration took place. The stumps did not curve at all, when the uppermost ring was cut off and placed unilaterally on them. This experiment was repeated after the tips had remained 9 hours on the stumps and then a slight regeneration did take place. When the coleoptile rings were placed unilaterally on the stumps these curved away from the ring. Of 40 plants 18 curved negatively, 2 positively and 20 remained straight. If we compare these curvatures with those of the first experiment it is obvious that under these circumstances the regenerated tip produces far less growth-accelerating substances than if the tip had not been replaced on the stump.

In order to determine the exact moment at which the new physiological tip appears, it was necessary to measure the growth of coleoptiles after decapitation.

The increase in length was measured every hour by means of a cathetometer. Fig. 1 demonstrates the average growth of 18 decapitated plants.

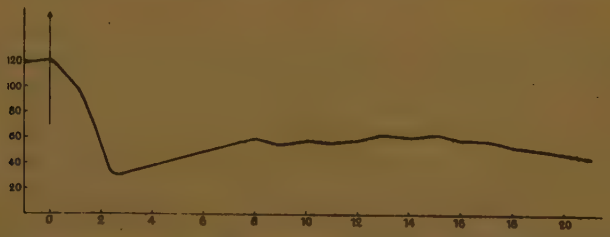

Fig. 1. Growthresponse after decapitation. At ↑ the plants were decapitated. The abscissa represents time in hours, the ordinate growth in 10 μ per hour. Temp. 21° C.
The average growth of 18 coleoptiles.

This curve shows that just after the decapitation the growth decreases slowly, then more rapidly and after 120—180 minutes there is a marked

minimum. The growth then increases steadily, but never reaches its former intensity. The growth of some decapitated coleoptiles was registered automatically by the auxanometer of KONINGSBERGER (2) in order to get exact data for the minimum. It turned out that the minimum was reached 150 minutes after the decapitation.

The growth of the plants on which the tips were replaced was also measured by means of the cathetometer.

It was found that the growth decreased gradually and in about 9 hours a minimum was reached.

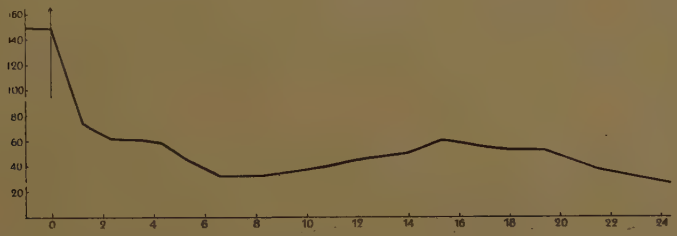

Fig. 2. Growthresponse of coleoptiles, upon which the tip was replaced. At ↑ the tip was cut off and placed on the stump. The abscissa represents time in hours, the ordinate rate of growth in 10 μ per hour. Temperature 21° C. Average growth of 15 coleoptiles.

If we compare this curve with that of the decapitated plants, then we find that the minimum, appearing 150 minutes after decapitation, is not due to a transitory effect of the wound.

The following experiment shows that 180 minutes after decapitation the uppermost zone of the coleoptile produces growth-accelerating substances which it did not do before.

TABLE I.

Number of coleoptiles	Decapitated at:	Upper most part: placed unilaterally	Result at:	+	−	Straight
3	0 h. p.m.	5.30 p. m.	9 h. p.m.	0	2	1
4	2.30 h. p.m.	„	„	0	3	1
3	4 h. p.m.	„	„	0	0	3
3	5.30 h. p.m.	„	„	0	1 (sl)	2

In order to examine when the geotropical sensibility returns a number of coleoptiles were decapitated and then placed horizontally. It turned out that the curvatures showed quite a different progress from the normal geotropical curvatures.

The curvatures appeared first in the base of the coleoptile and then moved to the tip. They were probably caused by the slight geotropical sensibility of the base.

On the other hand the coleoptiles curved quite normally when they were placed horizontally six hours after the decapitation. First the tip of the coleoptile curved, then the base while the upperpart straightened. It is apparent from the above described experiments that regeneration takes place here.

Some of these coleoptiles were decapitated for the second time before placing them horizontally. These coleoptiles curved in the base only, just as those placed horizontally immediately after the first decapitation. It was impossible to determine exactly the moment at which the geotropical sensibility returns, because the perception of the base could not be excluded. It is not difficult to exclude this perception with unilateral illumination. I illuminated only 2 mm of the tip of the coleoptile by using small screens of tin.

So it is much easier to determine the moment at which the phototropical sensibility returns. According to ROTHERT (5) the phototropical sensibility should return 3—6 hours after decapitation. These data are not exact as ROTHERT has not excluded the perception of the base. It is obvious that if a new tip regenerated in the uppermost part of a decapitated coleoptile this zone will become more sensitive to light. It was possible to illuminate exactly the same zone (3—4 mm below the tip) on intact coleoptiles by means of small tubes in which a narrow slit was made.

According to ARISZ (1) the maximum for the first positive curvature can be obtained by a quantity of light of about 250 Lux. I illuminated unilaterally decapitated and intact coleoptiles during 20 sec. with an intensity of 12.5 M.C. No curvatures were visible. The new physiological tip produces comparatively little growth-accelerating substances, therefore a reaction is obtained sooner by a long illumination with a small intensity than by a short one with a high intensity.

The intact coleoptiles did not curve at all after illuminating the zone 3—5 mm below the tip during 60 min. with 3 M.C. This agrees with the results of WILSCHKE (8) who determined the threshold at 20000 Lux. A number of coleoptiles were decapitated and a few hours afterwards the

TABLE II.

Temperature 21°. Relative moisture 60—70 %.

Number of coleoptiles	Decapitated at:	Illuminated at (3 M.C.)	Result at:	+	—	Straight	Positively
37	$11.^{50}$ h. a. m.	$0.^{20}$-$1.^{20}$ h. p. m.	$3.^{10}$ h. p. m.	3	0	34	8 %
36	$2.^{15}$ h. p. m.	$3.^{15}$-$4.^{15}$ h. p. m.	$5.^{50}$ h. p. m.	2	0	34	6 %
47	$10.^{50}$ h. a. m.	$0.^{20}$-$1.^{20}$ h. p. m.	$3.^{10}$ h. p. m.	9	0	38	18 %
58	$11.^{45}$ h. a. m.	$2.^{15}$-$3.^{15}$ h. p. m.	$5.^{15}$ h. p. m.	41	0	17	71 %
83	3 h. p. m.	$7.^{15}$-$8.^{15}$ h. p. m.	$9.^{15}$ h, p. m.	67	0	16	81 %

uppermost zone (2 mm) was illuminated unilaterally with the same quantity of light. One hour after illuminating all the coleoptiles curved towards the illuminated side. This experiment shows that after decapitation the uppermost zone of the coleoptile is more sensitive to light than it was before. In order to determine at which moment this sensibility appears, the coleoptiles were illuminated at different intervals after decapitation.

It is apparent from Table II that 150 min. after decapitation the uppermost zone has become sensitive to a quantity of light, which is not the case with the same zone of the intact coleoptiles.

Comparing this with the result of the growth measurements we see that there is a complete agreement between them. The growth acceleration also appears after 150 minutes. Then growth accelerating substances are produced again by the uppermost zone whereas the same zone of the intact coleoptile does not do so. These experiments very strongly favour the conception that light immediately acts upon the production of the growth accelerating substances.

Therefore we have to assume that photochemically light increases or retards a process which may also take place in the dark.

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Botany. — *"Phototropical curvatures of seedlings of Avena which appear when reaction of the distal side is excluded."* By H. RAMAER. (Communicated by Prof. F. A. F. C. WENT.)

(Communicated at the meeting of June 26, 1926).

The phototropical curvatures which appear when seedlings of Avena are illuminated unilaterally with different quantities of light, have been carefully studied by ARISZ (1). He found that a positive curvature appears with 1.4—4000 M.C.S., a negative curvature with 4000—70000 to 120000 M.C.S. and a second positive curvature with still greater quantities of light.

It has been tried in several different ways to explain the phototropical curvatures by a difference in the rate of growth between the illuminated and unilluminated sides of the coleoptile.

BOYSEN JENSEN (2) thought that he could conclude from his investigations about the locality of the conduction of the phototropical stimulus that the positive curvature is the result of an increase in the rate of growth of the unilluminated side. This increase would be induced by substances, formed, owing to the unilateral illumination, in the unilluminated side of the tip and diffusing through the distal side to the growing zone.

PAÁL (3) showed that Coix under normal circumstances always forms growth-accelerating substances in the tip and he explained the positive curvature by a decrease in the rate of growth of the illuminated side as a result of a onesided decrease in the production of growth-accelerating substances.

BLAAUW (4) sought connection between the phototropical curvatures and the lightgrowth-responses studied by him.

According to his theory, every part of a phototropically sensitive organ is able to show a growth-response to the quantity of light it receives, so that a phototropical curvature may be the result of the combinations of these growth-responses.

Bij means of this theory VAN DE SANDE BAKHUYZEN (5), later VAN DILLEWIJN (6) more accurately, could wholly explain the curvatures determined by ARISZ.

In consequence of a recent publication of BOYSEN JENSEN and NIELSEN (7), in which they again bring forward arguments in favour of the theory of BOYSEN JENSEN, I want to compare the three theories with each other and discuss them in connection with some recent investigations of C. VAN DILLEWIJN and F. W. WENT and with the results I obtained by the method of BOYSEN JENSEN and NIELSEN.

In their first experiment they made a median section in the tip of a coleoptile and placed a little screen in the cut. Round the base they placed

a small tube and then illuminated one half of the tip with an intensity of 25 M.C. for an indefinite period (2—3 hours). They found no curvature, but when the light fell in the direction of the screen a curvature appeared. Evidently the wound had no effect on the sensibility of the tip.

This result was an argument in favour of the theory of BOYSEN JENSEN. The same can be said of the following. The tip of a coleoptile was cut off and replaced by two sidely placed tips, between which a small screen was put. When one of the tips was illuminated unilaterally ($2\frac{1}{2}$ hours $\times$ 25 M.C.) the base showed a negative curvature which necessarily was the result of an increase in growth of the proximal side induced by the distal side of the illuminated tip.

I repeated both experiments, although chiefly the first, because the second can afford too many mistakes, for instance individual differences in the production of growth-accelerating substances, the one tip perhaps drying up more rapidly than the other, the contact with the base being unequal etc.

The tip was cut into two by putting a needle through the plant a few m.m. below the tip and then moving it upwards. With a little practice the tip can be accurately split into two in this way. In the cut a little black screen was placed.

The results with onesided illumination were as follows:

1. After onesided continuous illumination of the one half of the tip with 25 M.C. for four hours, the base shows a positive curvature.

This curvature is very slight which may be due to the action of the wound, because plants with a cut but without a screen and illuminated for the same time show the slight curvature while intact plants, as a control experiment, show a maximal curvature in the direction of the light.

As working with continuous illumination appeared to me to be an objection, because so little is known about the allsided continuous illumination, I decided to illuminate with certain definite quantities of light.

2. An allsided illumination with 800 M.C.S. causes a decrease in growth.

If I illuminated the plants unilaterally with 800 M.C.S. after they had been kept in the dark for five hours after the operation (in order to loose the effect of the wound) a positive curvature appeared which reached its maximum after two hours (fig. 1).

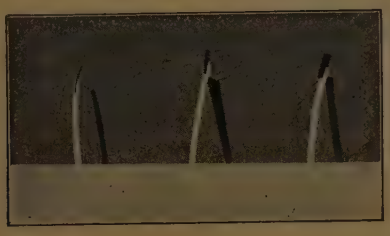

Fig. 1. Curvature after illuminating with 800 M.C.S. (10 sec. $\times$ 80 M.C.) from the right hand side. By means of a little screen light is prevented from reaching the base.

This curvature was shown by 80 % of the plants while 20 % remained straight.

From this result a few very important conclusions may be drawn.

First, that one half of the coleoptile is in itself able to show a lightgrowth-response, which corresponds with the decrease in growth, found by VAN DILLEWIJN, after allsided illumination with 800 M.C.S. and therefore wholly in accordance with the theory of BLAAUW.

Further, that PAÁL was right when he assumed this positive curvature to be the result of a decrease in growth of the proximal side. I must draw attention here to a recent research by WENT (8) who showed convincingly that the production of growth-accelerating substances in the tip decreases considerably with quantities of light of 800 M.C.S.

Finally the theory of BOYSEN JENSEN has been shown to fail in the area of the first positive curvature.

Besides, how could it be possible, as the latter assumes, that the distal side shows an increase in growth when the tip has been illuminated unilaterally with 800 M.C.S.? In this case illumination of both sides, and also allsided illumination with 800 M.C.S. would have to cause an increase in growth of the whole coleoptile which is a flat contradiction of the facts.

3. With his last experiments VAN DILLEWIJN (9) found an increase in growth with 80000 M.C.S.

At the same time WENT found that with 100000 M.C.S. the production of growth-accelerating substances increases considerably in the tip.

When I illuminated the seedlings which had been treated in the above mentioned manner, unilaterally with 80000 M.C.S., 90 % showed a negative curvature after one hour (fig. 2).

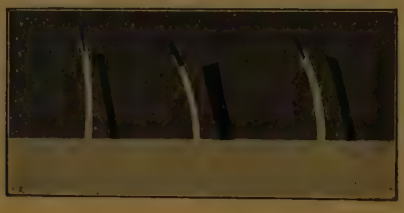

Fig. 2. Curvature after illuminating with 80000 M.C.S. (100 sec. × 800 M.C.) from the right hand side.

This time corresponds with that after which takes place an increase in growth after allsided illumination with 80000 M.C.S.

When we discuss shortly the acquired state of affairs then it is proved here that when a coleoptile is illuminated unilaterally with 800 M.C.S. and reaction of the distal side is excluded, the first positive curvature appears through a decrease in growth of the illuminated side.

This corresponds with the theory of PAÁL but not with that of BOYSEN

JENSEN, while the results are wholly in accordance with the theory of BLAAUW.

Along three different lines, namely by the lightgrowth-responses, by the research about the production of tipsubstances as well as by the above described experiments it has now been proved that the first positive curvature is the result of a decrease in growth of the illuminated side.

Besides, in the same way it has conclusively shown that with 80000 M.C.S. one half of the coleoptile independent of the other half may give an increase in growth.

Are there circumstances under which the distal side of coleoptile shows an increase in growth, while at he same time the proximal side grows in such a way, that a positive curvature results ?

VAN DILLEWIJN has shown that these circumstances are indeed realized in the area of the second positive curvature.

From this it is therefore evident that the theory of BOYSEN JENSEN is valid for the second positive curvature.

So each of the theories of PAÁL and BOYSEN JENSEN has been proved to be correct for a certain area, while both now fit in the theory of BLAAUW, which explains the phototropical curvatures by the difference in lightgrowth-response of the proximal and distal sides.

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Mathematics. — *„Zur Entstehung meiner Arbeiten über Dimensions- und Kurventheorie."* By KARL MENGER. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of May 29, 1926).

Im Folgenden werden (einem von mehreren Seiten geäusserten Wunsch zur Klärung von Prioritätsfragen entsprechend) einige Akten betreffend die Entstehung meiner dimensions- und kurventheoretischen Untersuchungen abgedruckt.

Bloss erwähnt werde ein im Juni 1921 Herrn Professor HAHN im Zusammenhang mit seinem damaligen punktmengentheoretischen Seminar übergebener Aufsatz, betitelt *„Der Begriff der Kurve"* [1]), in welchem im wesentlichen jene Gebilde eingeführt wurden, die ich später als reguläre Kurven bezeichnet habe (d.s. jene Kontinua, zu deren sämtlichen Punkten beliebig kleine Umgebungen mit endlichen Begrenzungen existieren), und in dem ferner End- und Knotenpunkte solcher Kurven definiert wurden.

Im Herbst 1921 hinterlegte ich bei der Wiener Akademie der Wissenschaften [2]) ein kurzes Manuskript *„Zur Theorie der Punktmengen"* [3]), welches hierunter nach einer notariell beglaubigten Abschrift abgedruckt wird:

Auf die Dimensionalität wird in der Theorie der Punktmengen im allgemeinen nur durch Zugrundelegung eines Raumes von bestimmter Dimensionszahl Rücksicht genommen. Es ist indes klar, dass, ganz unabhängig von dieser Zahl, von einer Dimensionalität gewisser Kontinua an sich gesprochen werden kann. Im folgenden Entwurf zu einer Abhandlung suche ich diesen Begriff zu präzisieren. Wir werden zu einer rekursiven Definition der Dimensionen gelangen und beginnen mit den eindimensionalen Kontinuen.

§ 1. Der Kurvenbegriff.

Der populäre Kurvenbegr. (nichtwissenschaftlichen Zwecken entsprungen) ist vage und nicht einheitlich. Wir suchen einen Begriff, der möglichst

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[1]) Das noch vorhandene Manuskript trägt den Vermerk: *„Eingegangen Juni 1921. H. HAHN".* — Sämtliche in dieser Note als noch vorhanden bezeichnete Manuskripte befinden sich gegenwärtig bei Herrn L. E. J. BROUWER.

[2]) Vgl. Anzeiger d. Wiener Akad. d. Wissensch. 58, 1921, S. 224.

[3]) Das noch vorhandene Manuskript wurde nach Eröffnung des Schreibens mit der Aufschrift versehen: *„Dieses Blatt bildete den Inhalt des zur Wahrung der Priorität der Akademie der Wissenschaften übergebenen versiegelten Schreibens No. 778, 1921. — 9. April 1926.* F. BECKE, *Generalsekretär der Akademie d. Wissenschaften in Wien".*

viele Gebilde in sich fasst, die gemeinhin als Kurven bezeichnet zu werden pflegen und (von notwendigen Verallgemeinerungen abgesehn) möglichst wenige Gebilde, die gemeinhin nicht als Kurven gelten würden. Dass der Jordansche Kurvenbegriff diesen Forderungen nicht genügt, ist ohne weiteres klar, da er auch Flächen, ja n-dimensionale Räume umfasst (Peano-„Kurven"). Lennes anderseits beschränkt sich von vorneherein auf die Definition „einfacher" Kurvenbögen.

Wir definieren: Ein Kontinuum $\mathfrak{K}$ heisst Kurve, wenn in jeder Umgebung $\mathfrak{U}$ jedes der Punkte P von $\mathfrak{K}$ eine Umgebung $\mathfrak{U}_1$ enthalten ist, so dass der Durchschnitt $\mathfrak{D}$ von $\mathfrak{K}$ mit der Begrenzung von $\mathfrak{U}_1$ keinen zusammenhängenden Teil enthält.

§ 2. Sätze über Kurven.

1. $\mathfrak{D}$ kann nicht für alle Punkte von $\mathfrak{K}$ aus einem einzigen Punkt bestehn.

2. Wenn $\mathfrak{D}$ für jeden P aus höchstens zwei Punkten bestehen soll, so gibt es höchstens zwei Punkte („End"-punkte von $\mathfrak{K}$) in denen $\mathfrak{D}$ nur aus einem einzigen Punkt besteht.

3. Wenn $\mathfrak{D}$ für alle P aus genau zwei Punkten bestehn soll, so ist $\mathfrak{K}$ entweder eine geschlossene oder eine nicht kompakte offene Kurve.

4. Der Lennesbogen ist eine Kurve. — Zu einfachen Kurven geht man durch ein Studium der Zahl und Natur der „Knoten"- und „End"-punkte von $\mathfrak{K}$ über (jener Punkte, für die $\mathfrak{D}$ aus mehr bezw. weniger als zwei Punkten besteht).

5. In $\mathfrak{K}$ gibt es keinen Teil, der umkehrbar eindeutiges und stetiges Abbild einer Kreisfläche wäre. Insbes. enthält also $\mathfrak{K}$ im $\mathfrak{R}_2$ keinen inneren Punkt.

§ 3. Der Flächenbegriff.

Ein Kontinuum $\mathfrak{F}$ heisst Fläche, wenn in jeder Umgebung $\mathfrak{U}$ jedes Punktes P von $\mathfrak{F}$ eine Umgebung $\mathfrak{U}_1$ enthalten ist, so dass der Durchschnitt $\mathfrak{D}$ von $\mathfrak{F}$ mit der Begrenzung von $\mathfrak{U}_1$ eindimensional ist, d.h. aus diskreten Kurven besteht.

(Dieses „diskret" lässt sich präzisieren. — Dann Sätze über Flächen u.s.w.)

§ 4. n-dimensional heisst ein Kontinuum, wenn $\mathfrak{D}$ (n—1)-dimensional ist.

Die Definitionen von § 1, 3, 4 habe ich (angeregt durch Herrn Prof. Hahn's Seminar über den Kurvenbegriff) bereits im April 1921 gefunden, wie Herr Prof. Hahn und Herr Otto Schreier bestätigen können. Die Sätze, 1, 2, 3 von § 2 habe ich auf Anregung Herrn Prof. Hahn bewiesen. Er war es auch der im § 1 die Worte „keinen zusammenhängenden Teil" an Stelle meiner ursprünglichen „höchstens abzählbar viele Punkte" gesetzt hat.

Ende des Jahres 1921 übergab ich Herrn HAHN als Herausgeber der „Monatshefte für Mathematik und Physik" den in der vorangehenden Note erwähnten Aufsatz zur Dimensions- und Kurventheorie. Ein unwesentliches Versehen in demselben veranlasste mich im Februar 1922 zur Einsendung einer Ergänzung folgenden Wortlautes [1]):

Eine nicht leere Menge M des R_m heisst n-dimensional, wenn

1. *zu jedem ihrer Punkte P und jeder offenen Umgebung $U_1(P)$ eine offene Umgebung $U_2(P) < U_1(P)$ existiert, mit deren Begrenzung M eine höchstens (n-1)-dimensionale Menge gemein hat; und wenn M*

2. *mindestens einen Punkt Q enthält, für den eine offene Umgebung U existiert derart, dass M mit der Begrenzung jeder offenen Umgebung $U_1(Q) < U$ eine Menge gemein hat, die einen (n-1)-dimensionalen Teil enthält. (—1)-dimensional ist die leere Menge.*

Genau (ev. „homogen") n-dimensional heisse eine Menge, die nur Punkte Q mit der Eigenschaft 2.) enthält. (Jede nulldimensionale Menge ist genau nulldimensional). Jeder Teil einer n-dimensionalen Menge ist höchstens n-dimensional. n-dimensionaler Kern einer n-dimensionalen Menge M heisse die Vereinigung aller genau n-dimensionalen Teile von M.

Diese Definition des allgemeinen Dimensionsbegriffes ist äquivalent und auch dem Wortlaute nach fast völlig übereinstimmend mit der in den Monatsheften für Mathematik und Physik Bd. 33 (S. 157 f.) enthaltenen Fassung.

Auf ein ausführliches im November 1922 Herrn HAHN eingesandtes Manuskript über Dimensions- und Kurventheorie werde ich gelegentlich näher eingehen.

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[1]) Das noch vorhandene Manuskript trägt den Vermerk *„Eingegangen unmittelbar nach Absendung (12. II. 1922); seither stets in meiner Verwahrung; an Herrn Dr. Menger ausgefolgt am 10. 4. 1926.* H. HAHN".

Mathematics. — *"Allgemeine Räume und Cartesische Räume"*. Zweite Mitteilung: *„Ueber umfassendste n-dimensionale Mengen"*. By KARL MENGER. (Communicated by Prof. L. E. J. BROUWER).

(Communicated at the meeting of June 26, 1926).

Wenn die n-dimensionale Menge N zu jedem n-dimensionalen kompakten metrischen Raum eine homöomorphe Menge als Teil enthält, dann wollen wir N eine *umfassendste n-dimensionale Menge* nennen. Wir weisen zunächst nach:

Es existieren umfassendste eindimensionale Mengen, darunter sogar beschränkte, stetig durchlaufbare Kurven des R_3. Wir können eine solche Menge in folgender Weise konstruieren: Der Einheitswürfel des R_3 wird in 27 homothetische, einander kongruente Teilwürfel von der Seitenlänge $\frac{1}{3}$ zerlegt. Sodann tilgen wir (nebst ihren Oberflächen) den innersten dieser Würfel (d.h. jenen, dessen Oberfläche zu der des Einheitswürfels fremd ist) und die sechs Würfel, welche mit dem innersten eine Fläche gemein haben. Zu jedem der zwanzig nicht getilgten Würfel nehmen wir seine Begrenzung, soweit sie getilgt wurde, wieder hinzu und wiederholen in ihm denselben Vorgang der Zerlegung in 27 Teilwürfel und der Tilgung der sieben mittleren. Indem man so fortfährt, erhält man beim n-ten Schritt 20^n Würfel von einer Seitenlänge $\frac{1}{3^n}$, welche welche wir „Würfel des n-ten Schrittes von I" nennen und deren Summe wir mit I_n bezeichnen. Die (offenbar eindimensionale stetig durchlaufbare) Menge $\prod_{n=1}^{\infty} I_n$ ist unsere umfassendste eindimensionale Menge I.

Man kann I auch in folgender Weise darstellen: Die Summe der Kanten aller Würfel des n-ten Schrittes von I bezeichnen wir als G_n und nennen sie das n-te Gerüst von I. Der Raum I ist dann die abgeschlossene Hülle seines Gerüstes $G = \sum_{n=1}^{\infty} G_n$.

Der Nachweis der Einbettbarkeit jedes eindimensionalen kompakten metrischen Raumes A in den R_3 wurde in der ersten Mitteilung [1]) durch

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[1]) Vgl. diese Proceedings, Bd. **29**, 1926, S. 496. Es wurde daselbst das einfache, aber zahlreicher Anwendungen fähige Lemma bewiesen: Die Kerne homologer finiter Umgebungssysteme sind homöomorph. Die Umkehrung, dass nämlich in homöomorphen kompakten Räumen homologe finite Umgebungssysteme existieren, ist trivial. *Notwendig und hinreichend für die Homöomorphie zweier kompakter Räume ist also die Existenz homologer finiter Umgebungssysteme in ihnen.* Sind zwei Räume als Kerne von finiten Umgebungssystemen gegeben, so ist die Frage nach der Homöomorphie der beiden Räume zurückgeführt auf die Frage, ob aus den beiden definierenden Umgebungssystemen zwei homologe Umgebungssysteme hergeleitet werden können, und das ist eine Frage nach der Möglichkeit einer abzählbaren Folge von finiten Operationen.

die Angabe eines finiten Umgebungssystems in A und eines dazu homologen Systems von Polyedern des R_3 geführt. Bezeichnen wir alle Würfel irgend eines n-ten Schrittes von I kurz als *Würfel von* I und ein Polyeder, welches Summe endlich vieler Würfel von I ist, als *Intervallsystem von* I, so haben wir, um die Einbettbarkeit von A in die Menge I nachzuweisen, *ein finites Umgebungssystem in* A *und ein homologes System von Intervallsystemen von* I *anzugeben.*

Setzen wir nun voraus: Es sei A eine kompakte Menge und es seien $B_1, B_2, \ldots B_m$ m paarweise fremde Teilmengen von A. Es sei ferner P ein Intervallsystem von I und es seien $Q_1, Q_2, \ldots Q_m$ m paarweise fremde Quadratflächen auf der Oberfläche von P. Es sei endlich A Summe von n abgeschlossenen Umgebungen $A_1, A_2 \ldots A_n$, die zu je zweien keine inneren Punkte gemein haben und so dass jeder Punkt von A höchstens zwei Mengen A_i, jeder Punkt einer Menge B_k genau einer Menge A_i angehört. Auf Grund der Tatsache, dass für jedes n^{te} Gerüst G_n von I die Menge $G_n - G_{n-1}$ zusammenhängend ist, zeigt man durch elementargeometrische Ueberlegungen, dass unter den angegebenen Voraussetzungen folgendes möglich ist: Wählt man im Innern von P irgend n verschiedene Punkte $p_1, p_2, \ldots, p_n$ von P, dann kann man so kleine paarweise fremde Würfel von I $W_1, W_2, \ldots W_n$ angeben, dass p_i im Innern von W_i liegt und dass sich aus dem Gerüst von I paarweise fremde Streckenzüge angeben lassen, welche die Oberfläche jedes W_i mit den Oberflächen aller übrigen W_k und mit allen Q_k verbinden. Daraus folgt aber sofort die Existenz von Intervallsystemen von I $P_1, P_2, \ldots P_n$, sodass alle $P_i < P$ sind und dass (für jedes i und k) P_i mit P_k, bzw. mit Q_k dann und nur dann Punkte u. zw. genau eine Quadratfläche gemein hat, wenn A_i mit A_k, bzw. mit B_k Punkte gemein hat. Damit sind in Hilfssatz 1 der ersten Mitteilung der Polyeder speziell durch Intervallsysteme von I ersetzt.

Hilfssatz 2 der ersten Mitteilung bedarf keiner Verschärfung. Die Verschärfung von Hilfssatz 3 besagt, dass man zu einem vorgelegten kompakten eindimensionalen Raum A eine bestimmt geartete Zerlegung in beliebig kleine Umgebungen und gleichzeitig in einem vorgelegten Intervallsystem P von I ein entsprechendes System von beliebig kleinen Intervallsystemen von I unter Erfüllung beiderseitiger Randbedingungen angeben kann. Um dies einzusehen, hat, man ganz wie beim Beweis von Hilfssatz 3 zunächst *irgend* eine Zerlegung von A mit den gewünschten Eigenschaften in Teile von gewünschter Kleinheit anzugeben und kann zu dieser Zerlegung nach dem verschärften Hilfssatz 1 entsprechende Intervallsysteme von I in P angeben, welche auch die entsprechenden Randbedingungen erfüllen. Haben diese Intervallsysteme auch bereits die gewünschte *Kleinheit*, dann ist man am Ziel. Andernfalls ersetzt man jedes der zu grossen Intervallsysteme durch eine Kette von Intervallsystemen von I mit den Eigenschaften, die in Hilfssatz 2 vorausgesetzt werden, und bestimmt zu diesen Polyedern nach Hilfssatz 2

Teilumgebungen der Umgebung in A in entsprechender Lage. Damit ist die Verschärfung von Hilfssatz 3 bewiesen, auf Grund derer ein finites Umgebungssystem im vorgelegten eindimensionalen Raum und ein homologes System von Intervallsystemen von I konstruiert werden kann.

Aus diesen Überlegungen gehen zugleich die beiden folgenden Nebenresultate hervor: *Jeder eindimensionale kompakte metrische Raum ist homöomorph mit einer Menge des R_3, zu deren sämtlichen Punkten beliebig kleine Intervalle existieren, mit deren Begrenzungen die Menge diskontinuierliche Durchschnitte hat.* Und: *Jeder kompakte eindimensionale Raum ist homöomorph mit einem Teil einer Fläche des R_3.* Wir können nämlich, so wie bei der Konstruktion der Menge I, den Würfel in 27 Teile teilen, aber bloss den innersten dieser Würfel ohne seine Begrenzung tilgen und sodann dieses Verfahren in den 26 nicht getilgten Würfeln wiederholen und so ad infinitum fortfahren. Nennen wir II_n die Summe aller 26^n nicht getilgten Würfel von der Seitenlänge $\frac{1}{3^n}$, und bezeichnen wir mit II den Durchschnitt aller Würfelsummen II_n, dann ist die so entstehende Menge offenbar eine (im kleinen zusammenhängende) Fläche, welche die Menge I als Teil enthält[1]).

Von Interesse ist die Untersuchung dieser Verhältnisse in höheren Dimensionen. Der n-dimensionale Einheitswürfel kann in 3^n homothetische, einander kongruente Teilwürfel von der Seitenlänge $\frac{1}{3}$ zerlegt werden. Jeder dieser Würfel lässt sich charakterisieren durch ein n-Tupel gebildet aus den Zahlen 0, 1, 2. indem man jedem Würfel als k-te Koordinate seinen Abstand von der $(n-1)$-dimensionalen Hyperebene durch den Ursprung, welche auf der k-ten Achse senkrecht steht, zuordnet. Es befinden sich unter den 3^n Würfeln $\binom{n}{n-k} \cdot 2^{n-k}$, unter deren Koordinaten genau k Einsen vorkommen. Wir tilgen nun samt ihren Oberflächen jene Würfel, unter deren Koordinaten mindestens $m+1$ Einsen vorkommen. Wir behalten $\mu(m) = \sum_{k=0}^{m} \binom{n}{n-k} \cdot 2^{n-k}$ Würfel zurück, nehmen zu jedem die Oberfläche, soweit sie getilgt wurde, wieder hinzu und wiederholen in jedem dieser Würfel dasselbe Verfahren der Unterteilung und Tilgung der $\sum_{k=m+1}^{n} \binom{n}{n-k} \cdot 2^{n-k}$ mittleren Würfel. Beim k-ten Schritt behalten wir $\mu(m$ Würfel von einer Seitenlänge $\frac{1}{3^k}$, deren Summe wir mit ${}_kR_n^m$ bezeichnen.

[1]) Man kann die Mengen, welche umfassendste eindimensionale Mengen sind, in verschiedener Weise charakterisieren: dadurch, dass sie einen mit der Menge I homöomorphen Teil enthalten, — oder dadurch, dass sie einen Teil enthalten, in dessen sämtlichen offenen Teilen (vermöge ihrer Zusammenhangsverhältnisse) die Verkettungen endlich vieler Teilumgebungen durchgeführt werden können, die beim Beweis der Einbettbarkeit aller eindimensionalen Mengen in die Menge I verwendet wurden, — oder als abgeschlossene Hüllen eines hinreichend verzweigten Gerüstes u.s.w.

Die Menge $\prod_{k=1}^{\infty} {}_kR_n^m$, von der man leicht zeigt, dass sie stetig durchlaufbar und m-dimensional ist, nennen wir R_n^m.[1])

In Verallgemeinerung einer Schlussweise von SIERPIŃSKI[2]) zeigt man unschwer, dass für jedes n die Menge R_n^{n-1} zu jeder nirgends dichten abgeschlossenen Teilmenge des R_n, also zu jeder höchstens $(n-1)$-dimensionalen abgeschlossenen Menge des R_n, eine homöomorphe Menge als Teil enthält. Es gilt aber viel mehr: es zeigt sich nämlich, *dass die Menge* R_n^{n-1} *ein topologisches Bild von jeder* (*auch nicht-abgeschlossenen*) *weniger als n-dimensionalen Menge des* R_n *als Teil enthält.* Nach einem Satz des Dimensionstheorie ist jede weniger als n-dimensionale Menge des R_n enthalten in einer Menge, deren Komplement abzählbar und im R_n dicht ist. Einem bekannten Satz von FRÉCHET zufolge sind alle Mengen des letzteren Art untereinander homöomorph. Es genügt also zu zeigen, dass ein Teil der Menge R_n^{n-1} homöomorph ist mit einer Menge, deren Komplement abzählbar und im R_n dicht ist. Dies trifft aber, wie man nachweisen kann, für die G_δ-Menge zu, welche entsteht, wenn man aus dem offenen Einheitswürfel von seinen 27 Teilwürfeln den innersten *nebst seiner Begrenzung* tilgt und dieses Verfahren in jedem der übrigen 26 Teilwürfel und so ad infinitum fortsetzt.

Ich halte es für sehr wahrscheinlich, dass allgemein die Menge R_n^m eine umfassendste m-dimensionale Menge hinsichtlich des R_n ist, d.h. zu jeder m-dimensionalen Menge des R_n eine homöomorphe Menge als Teil enthält. Der Beweis dieses Satzes dürfte keine prinzipiellen Schwierigkeiten bieten, allerdings für $m < n-1$ recht langwierig sein.

In einer folgenden Mitteilung wird bewiesen werden, dass jeder n-dimensionale kompakte (und vermutlich sogar jeder n-dimensionale separable) metrische Raum homöomorph ist mit einer Teilmenge des R_{2n+1}. Ein stetig durchlaufbares Kontinuum, welches ein topologisches Bild jedes kompakten endlichdimensionalen Raumes als Teil enthält, ist dann z. B. die Menge $\sum_{n=1}^{\infty} I_n$, wobei I_n $(n=1, 2, \ldots)$ ein einen festen Punkt p enthaltendes n-dimensionales Intervall mit der Seitenlänge $\frac{1}{n}$ ist.

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[1]) R_1^0 ist die nirgends dichte perfekte Cantorsche Menge, R_2^1 die umfassendste ebene Kurve von SIERPIŃSKI (Comptes Rendus, 162, S. 629), R_3^1 die oben konstruierte umfassendste eindimensionale Menge.

[2]) a. a. O.

Mathematics. — *"Representation of a simplicial manifold on a locally simplicial manifold"*. By WILFRID WILSON. (Communicated by Prof. L. E. J. BROUWER).

(Communicated at the meeting of September 25, 1926).

The object of the following investigation is to prove the extension of the theorem I of BROUWER quoted beneath. The proof, of which the leading idea was suggested to me by Prof. BROUWER, consists in repeated applications of the methods and results of his three papers

"Ueber Abbildung von Mannigfaltigkeiten",

"Beweis des n-dimensionalen Jordanschen Satzes", and

"Jordansche Mannigfaltigkeiten",

in Bd. 71 of the Mathematische Annalen; and which for brevity will be quoted as "A. v. M.", "J. S." and "J. M." respectively.

1. By an n-dimensional element [1]) is understood the topological image of an n-dimensional simplex of the n-dimensional number space R_n.

By a simplicial n-dimensional manifold is understood one as defined by BROUWER [2]), that is, a connected set of points built up of n-dimensional elements any two of which have either no point in common, or only a p-dimensional face together with all its faces of lower dimensions in common $(0 \leqslant p \leqslant n-1)$, while the incidence relations of elements having a common vertex are those of the simplexes of a simplex star in R_n. The manifold is said to be closed or open according as the number of elements is finite or infinite.

By a locally simplicial n-dimensional manifold is understood a connected topological space [3]) to whose defining system of neighbourhoods there exists a countable [4]) equivalent [5]) system of neighbourhoods each of which is an n-dimensional element. The manifold is said to be closed or open according as it is or is not a compact topological space. It follows from this definition that if E' and E'' be two elements of a closed locally simplicial manifold, a chain $E_1, E_2, \ldots, E_m$ of elements of the manifold exists such that $E_i E_{i+1} \neq 0$, $E_1 = E'$ and $E_m = E''$, $(i = 1, 2, \ldots, m-1)$.

2. The elements of a locally simplicial manifold may be assumed. by a suitable subdivision, to be so chosen that any two having common

1) BROUWER, "A. v. M." p. 97.
2) BROUWER, "A. v. M." p. 97—98.
3) HAUSDORFF, "Grundzüge der Mengenlehre", p. 213.
4) HAUSDORFF's Grundzüge, Axiom (F), p. 263.
5) HAUSDORFF's Grundzüge, Kap. VIII, §, 1. 260.

points are both contained in a third element and thus constitute a two-sided piece of the manifold.

The indicatrix of an element Δ of a locally simplicial manifold being assigned, the indicatrix of an element E of a simplicial division of Δ is determined by the rule used for simplicial manifolds. Let now E be an arbitrary element contained in Δ, P any point in E, Δ' the representative simplex [6]) of Δ, E' and P' the images of E and P respectively in Δ', and B' the boundary of E'. The order of P' with respect to the Jordan Manifold B' in Δ' is ± 1 [7]) and the positive indicatrix of E is defined as that for which this order is $+1$. Consider now two elements E' and E'' such that $E'E'' \neq 0$ and a third element $E \subset E'E''$ [8]). The indicatrix of E' being assigned, that of E is determined as above so that the order of a point $P \subset E$ with respect to the boundary of E in E' [9]) is $+1$ and that of E'' is assigned so that the order of P with respect to the boundary of E in E'' [9]) is also $+1$. Finally let E' and E'' be arbitrary elements of the manifold and $\Delta_1, \Delta_2, \ldots, \Delta_m$ a chain of elements for which $\Delta_i \Delta_{i+1} \neq 0$, $\Delta_1 = E'$ and $\Delta_m = E''$. If then the indicatrix of E' be assigned, an indicatrix for E'' is determined by repeated applications of the above rule. When the indicatrix of E'' thus determined is independent of the chain $\Delta_1, \Delta_2, \ldots, \Delta_m$ of elements used in connecting E' and E'', the manifold is said to be two sided; otherwise it is called one sided.

It can now be proved that the following theorem of BROUWER [10]) —

I. If a closed, two-sided, simplicial, n-dimensional manifold μ be uniquely and continuously represented on a simplicial n-dimensional manifold μ', there exists a finite whole number c invariant under continuous modification of the representation, with the property that the image of μ covers every region of μ' altogether c times. If μ' be one sided or open c is always zero —

can be extended to the case where μ' is any locally simplicial n-dimensional manifold.

3. Let α be a unique continuous representation of the closed, two-sided, simplicial n-dimensional manifold μ on the locally simplicial n-dimensional manifold μ', and first consider the case where μ' is closed and two-sided. Then the simplicial division ζ of μ, the fundamental simplexes, faces and vertices of μ, ordinary fundamental simplexes of μ,

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6) BROUWER, "A. v. M.", p. 100.

7) BROUWER, "J. M.", p. 323.

8) Such an element E exists by HAUSDORFF's Axiom (B), Grundzüge p. 213.

9) By the order of a point P with respect to a JORDAN manifold in any element is meant the order of the image of P with respect to the image of the JORDAN manifold in the representative simplex of the element. The indicatrix of an element is related to that of its boundary by the rule given in BROUWER, "A. v. M.", p. 107.

We are here using BROUWER's generalized indicatrix. "J. M.", §§ 4 and 5.

10) "A. v. M", Satz 1, p. 106.

simplicial and modified simplicial representations β and γ respectively, corresponding to ζ and approximating to α, the "inner simplex" of each element of μ', the ordinary points of the inner simplexes of μ' and the numbers p and p' of positive and negative image simplexes respectively covering an ordinary point P of μ', as defined and used by BROUWER lead to the results [11] —

(1) The number $c = p - p'$ is constant in the ordinary points of an inner simplex of μ'

(2) c has the same value for every simplicial representation approximating to α.

4. Let E' and E'' be elements of the manifold μ' such that $E'E'' \neq 0$ and let the corresponding inner simplexes J' and J'' be such that $J'J'' \neq 0$. Consider a sequence $\zeta_1, \zeta_2, \ldots, \zeta_\lambda, \ldots, \zeta_\nu, \ldots, (\nu > \lambda)$, of simplicial divisions of μ such that ζ_{i+1} is obtained by simplicial divisions of the elements of ζ_i, the maximum diameter of any element of ζ_i being $< \frac{1}{2^i}$.

Let β'_λ and β''_λ be the simplicial approximations [12] to α in J' and J'' respectively corresponding to ζ_λ, γ'_λ and γ''_λ the corresponding modified simplicical approximations [13], P any ordinary point [14] of J' and J'' with respect to both γ'_λ and γ''_λ, and c'_λ and c''_λ the values of the number $p - p'$ at P for γ'_λ and γ''_λ respectively. It is required to prove that for sufficiently great λ, $c'_\lambda = c''_\lambda$.

Let E^λ be any image simplex under γ'_λ contained in $E'E''$, E the corresponding ordinary simplex of the subdivision ζ_λ of μ,

$E_1, E_2, \ldots, E_{\alpha_\nu}$ the simplexes of the subdivision ζ_ν of E,

$E^\lambda_1, E^\lambda_2, \ldots, E^\lambda_{\alpha_\nu}$ their images in $E^\lambda \subset E'$ under γ'_λ,

$\triangle^\nu_1, \triangle^\nu_2, \ldots, \triangle^\nu_{\alpha_\nu}$ their images in E'' under γ''_ν,

and $\triangle^\lambda_1, \triangle^\lambda_2, \ldots, \triangle^\lambda_{\alpha_\nu}$ those simplexes of E'' having the same vertices as $E^\lambda_1, E^\lambda_2, \ldots, E^\lambda_{\alpha_\nu}$ respectively. **.

The boundary B^ν of $\triangle^\lambda_1 + \triangle^\lambda_2 + \ldots + \triangle^\lambda_{\alpha_\nu}$ consists of those $(n-1)$-dimensional faces incident with only one of the $\triangle^\lambda_i$, and constitutes a simplicial image in E'' of the boundary B of E'. Let the number η be defined thus: if E^λ does not contain P, $\eta = 0$; if $E^\lambda \supset P$, $\eta = +1$ or -1 according as the indicatrix of E^λ is positive or negative in E'. Then the indicatrix of E'' being determined from that of E' by the rule given in § 2, the order of P with respect to the Jordan manifold B in E''* is η.

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[11] BROUWER, "A. v. M." pages 101—104.

[12] BROUWER, "A. v. M.", p. 101—102.

[13] " " " " p. 102.

[14] " " " " p. 103.

**) That is, the $\triangle_i$ are images in E'' of simplexes in the representative simplex of E''.

Therefore for sufficiently great ν any segment path in E'' [15]) from P to the boundary of E'', no vertex and no segment of which is in B^ν and no segment of which cuts an $(n-2)$-dimensional face of B^ν, crosses B^ν p times positively and p' times negatively where $p-p'=\eta$. [16]) Since an indicatrix of an $(n-1)$-dimensional face incident with two simplexes Δ_i^λ and Δ_j^λ is of different sign according to whether it is regarded as bounding Δ_i^λ or Δ_j^λ [17]), the above mentioned segment path also crosses the sum of the boundaries of the simplexes $\Delta_1^\lambda, \Delta_2^\lambda, \ldots, \Delta_{z_\lambda}^\lambda$, η times. If therefore the number of positive and negative of these simplexes containing P be q and q' respectively, we must have $q-q'=\eta$. Proceeding in this way with all the image simplexes such as E^λ it follows that $c_\lambda'=r-r'$ where r and r' are the numbers of positive and negative simplexes respectively covering P from all sets of type $\Delta_1^\lambda, \Delta_2^\lambda, \ldots, \Delta_{z_\lambda}^\lambda$. If now the vertices of each simplex Δ_i^λ be moved continuously into the corresponding vertices of Δ_i^ν, the number $r-r'$ remains constant if λ be sufficiently great, and is therefore equal to the corresponding number c_ν'' for the image simplexes Δ_i^ν under γ_ν'', that is, $c_\nu''=r-r'=c_\lambda'$. Therefore from (2) of § 3, $c_\nu'=c_\nu''$, and since any two elements of μ' may be joined by a finite chain of elements as in § 1, it follows that —

(3) The number c has the same value in the ordinary points of all inner simplexes of μ'.

As with (1) and (2), the theorems (4), and (5) beneath, follow exactly as by BROUWER [18]): —

(4) When the image of μ under α is not everywhere dense in μ', the degree c is zero.

(5) If μ' be one sided or open c is zero.

The above mentioned extension of I has thus been etablished. That this extension remains valid when μ is a closed, two-sided, n-dimensional pseudo-manifold, follows immediately from the definition of a pseudo-manifold [19]).

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*) See footnote 9).

15) By a segment path in E'' is meant the image in E'' of a segment path in the representative simplex of E''.

16) BROUWER, "Beweis des n-dimensionalen JORDANsche Satzes", Math. Ann. **71**, p. 317 footnote and J. M. § 4. p. 323.

17) BROUWER, "A. v. M.", p. 107.

18) A. v. M.", p. 106.

19) BROUWER, "Beweis der Invarianz des n-dimensionalen Gebiets", Math. Ann. Bd. **71**, p. 305—306.

ERRATUM.

In Vol. XXII of these Proceedings, on page 811 in the title of the paper: *"Ueber eineindeutige, stetige Transformationen von Flächen in sich"*, the words "Sechste Mitteilung" are to be replaced by "Siebente Mitteilung".

On page 814 of the same paper, the example given of a transformation of the first kind of degree n with a single invariant point, is irrelevant. Moreover, from footnote [2]) on page 865 of Vol. XXIX of these Proceedings, follows that by transformations of the first kind of positive degree, certainly at least two invariant points appear.

In Vol. XXIII of these Proceedings, on page 232 in the title of the paper: *"Ueber eineindeutige, stetige Transformationen von Flächen in sich"*, the words "Siebente Mitteilung" are to be replaced by "Achte Mitteilung".